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Introduction

How To Use This Guide

This guide is intended to **supplement** the syllabus readings. Although I believe it provides a thorough review of the exam material, the readings provide additional context that is invaluable. Please do NOT skip the syllabus readings.

Original Mathematical & Essay Problems

Original mathematical & essay problems/solutions are included for all papers. The original essay problems are my version of notecards. If a topic is covered in an essay problem, then you should know it. All original practice problems are included in the guide and as separate Excel workbooks. The Excel workbooks can be downloaded from the online course.

Past CAS Exam Problems

Past CAS exam problems & solutions are included for each paper. Note that these questions are solely owned by the CAS. They are included in the online course for student convenience. All past CAS problems are included in the guide and as separate Excel workbooks. The Excel workbooks can be downloaded from the online course.

Website

Outside of the occasional email, all study guide updates (errata updates, important dates, supplementary material, etc.) will be announced via the "News" page of the website. All study material (i.e. study guide, practice exams, online videos, supplementary workbooks, errata, etc.) can be found in the online course.

Questions

If you have a question about a particular topic in a paper or the study guide, feel free to shoot me an email at **michael@casualfellow.com**. I typically respond within 1-2 business days.

Errata

Although many hours were spent editing this study guide, errors are inevitable. As you notice them, please email me at **michael@casualfellow.com**. An errata sheet will be posted on the online course and will be updated on an as needed basis.

Blank Pages

Since many students want a printed copy of the study guide, blank pages have been inserted throughout the guide to ensure that all outlines start on odd pages.

Bookmarks

Bookmarks have been added for each section listed in the table of contents for easier navigation in Adobe Acrobat.

Mack (2000)

Outline

◇ Notation

- p_k is the proportion of the ultimate claims amount which is expected to be paid after k years of development
- $q_k = 1 - p_k$ is the proportion of the ultimate claims amount which is expected to remain unpaid after k years of development
- $U_0 = U^{(0)}$ is the a priori expectation of ultimate losses (i.e. expected ultimate losses)
- $U_{BF} = U^{(1)}$ is the Bornhuetter/Ferguson ultimate claims estimate
- $U_{GB} = U^{(2)}$ is the Gunner Benktander ultimate claims estimate
- $U_{CL} = U^{(\infty)}$ is the chain ladder ultimate claims estimate
- $U^{(m)}$ is the ultimate claim estimate at the m^{th} iteration
- U_c is a credibility weighted ultimate claims estimate, where c is the credibility factor
- $\hat{U}$ is any ultimate claims estimate
- R_{BF} is the Bornhuetter/Ferguson reserve estimate
- R_{CL} is the chain ladder reserve estimate
- R_{GB} is the Gunner Benktander reserve estimate
- $\hat{R}$ is any reserve estimate
- C_k is the actual claims amount paid after k years of development

◇ General relationship between any reserve estimate $\hat{R}$ and the corresponding ultimate claims estimate $\hat{U}$:

$$\boxed{\hat{U} = C_k + \hat{R}}$$

◇ **Bornhuetter/Ferguson method**

- Reserve estimate based on the a priori expectation of ultimates losses:

$$\boxed{R_{BF} = q_k U_0}$$

- Using the general relationship described earlier, $U_{BF} = C_k + R_{BF}$

- Since R_{BF} uses U_0, it assumes the current claims amount C_k is not predictive of future claims

◇ **Chain ladder method**

- $U_{CL} = C_k/p_k$
- Using the general relationship described earlier, $R_{CL} = U_{CL} - C_k$
- Combining the two previous formulae, it can be shown that

$$\boxed{R_{CL} = q_k U_{CL}}$$

- Since R_{CL} uses U_{CL}, it assumes the current claims amount C_k is fully predictive of future claims
- **Advantage of *CL* over *BF***: Using *CL*, different actuaries obtain similar results. This is not the case with *BF* due to differences in the selection of U_0

◇ **Benktander method**

- Also known as Iterated Bornhuetter/Ferguson method
- Since *CL* and *BF* represent extreme positions (fully believe C_k vs. do not believe at all), Benktander replaced U_0 with a credibility mixture:

$$\boxed{U_c = cU_{CL} + (1-c)U_0}$$

- As the claims C_k develop, credibility should increase. As a result, Benktander proposed setting $c = p_k$ and estimating the claims reserve by $R_{GB} = R_{BF} \cdot \frac{U_{p_k}}{U_0}$
- Combining this with the formula for R_{BF}, we can easily show that $R_{GB} = q_k U_{p_k}$
- Using our credibility mixture, we can show that $U_{p_k} = p_k U_{CL} + q_k U_0 = C_k + R_{BF} = U_{BF}$, which finally brings us to the following:

$$\boxed{R_{GB} = q_k U_{BF}}$$

- This equation has the following implications:
  - ◇ R_{GB} is obtained by applying the *BF* procedure twice, first to U_0, and then to U_{BF} (hence, the Iterated Bornhuetter/Ferguson method)
  - ◇ The Benktander method is a credibility weighted average of the *BF* method and the *CL* method, where $c = p_k = 1 - q_k$:

$$\boxed{\begin{aligned} U_{GB} &= C_k + R_{GB} \\ &= (1-q_k)U_{CL} + q_k U_{BF} \end{aligned}}$$

- Note: $U_{GB} = C_k + R_{GB} = (1 - q_k^2)U_{CL} + q_k^2 U_0 = U_{1-q_k^2} \neq U_{p_k}$, which illustrates the fact that the BF method and GB produce different results. It also shows that the Benktander method is a credibility weighted average of the CL method and the a priori expectation of ultimate losses, where $c = 1 - q_k^2$
- It is also possible to apply the credibility mixture directly to the reserves instead of the ultimates. Esa Hovinen proposed the following reserve estimate: $R_{EH} = cR_{CL} + (1-c)R_{BF}$. If we set $c = p_k$ as before, we find that $R_{EH} = R_{GB}$

◇ In his paper, Mack presents a theorem that shows how ultimates and reserves change as we iterate through indefinitely (rather than just iterating twice for the GB method). Since I don't think it's worth memorizing for the exam, let's just get to the results. Using the iteration rules $R^{(m)} = q_k U^{(m)}$ and $U^{(m+1)} = C_k + q_k U^{(m)}$, we obtain the following credibility mixtures:

$$\boxed{\begin{aligned} U^{(m)} &= (1 - q_k^m)U_{CL} + q_k^m U_0 \\ R^{(m)} &= (1 - q_k^m)R_{CL} + q_k^m R_{BF} \end{aligned}}$$

◇ If we iterate between reserves and ultimates indefinitely, we will eventually end up with the CL result

◇ The Benktander method is superior to BF and CL for a few reasons:

- **Lower mean squared error (MSE)**
  - ◇ Walter Neuhaus compared the MSE of $R_c = cR_{CL} + (1-c)R_{BF}$ for $c = 0$ (BF), $c = p_k$ (GB), and $c = c^*$ (optimal credibility reserve that minimizes the MSE)
  - ◇ MSE of R_{GB} is smaller than MSE of R_{BF} when $c^* > p_k/2$. This makes sense because the inequality implies that c^* is closer to $c = p_k$ than to $c = 0$
  - ◇ Mack also states in the abstract that the Benktander method almost always has a smaller MSE than BF and CL
- **Better approximation of the exact Bayesian procedure**
- **Superior to *CL* since it gives more weight to the a priori expectation of ultimate losses**
- **Superior to *BF* since it gives more weight to actual loss experience**

Original Mathematical Problems & Solutions

MP #1

Given the following information for accident year 2012 as of December 31, 2012:

- ◇ 12-ultimate cumulative paid LDF = 1.60
- ◇ Ultimate loss based on the chain-ladder method = $12,000
- ◇ Ultimate loss based on the Benktander method = $14,000

Calculate the accident year 2012 ultimate loss based on the Bornhuetter/Ferguson method.

Solution:

- $U_{GB} = (1 - q_k)U_{CL} + q_k U_{BF}$
- $q_k = 1 - p_k = 1 - \frac{1}{\text{LDF}} = 1 - \frac{1}{1.6} = 0.375$
- Plugging q_k into our formula for U_{GB}, we have 14000 = (1 - 0.375)12000 + 0.375(U_{BF})
- Thus, $\boxed{U_{BF} = \$17{,}333.33}$

MP #2

Given the following:

	Cumulative Paid Losses ($)			
AY	12 mo.	24 mo.	36 mo.	48 mo.
2009	7,000	10,500	12,600	13,860
2010	8,000	12,000	14,400	
2011	9,000	13,500		
2012	10,000			

- ◇ The 2010 earned premium is $25,000
- ◇ The expected loss ratio for each year is 75%
- ◇ Assume the 48-ultimate loss development factor is 1.05

Calculate the accident year 2010 ultimate loss based on the Benktander method.

Solution:

- ◇ $U_{GB} = C_k + R_{GB}$
- ◇ From the loss triangle, $C_k = 14400$
- ◇ We need to calculate $R_{GB} = q_k U_{BF}$
- ◇ To determine q_k, we need to calculate the 36-ultimate LDF:
  - The 36-48 LDF is $13860/12600 = 1.10$
  - Combining this with the 48-ultimate LDF gives a 36-ultimate LDF of $(1.10)(1.05) = 1.155$
  - Then, $q_k = 1 - \frac{1}{1.155} = 0.134$
- ◇ To determine U_{BF}, we need to calculate U_0 for 2010:
  - $U_0 = EP \cdot ELR = 25000(0.75) = 18750$
  - $U_{BF} = C_k + R_{BF} = C_k + q_k U_0 = 14400 + 0.134(18750) = 16912.50$
- ◇ We can now calculate $R_{GB} = 0.134(16912.50) = 2266.275$
- ◇ Finally, $U_{GB} = 14400 + 2266.275 = \boxed{\$16{,}666.28}$

MP #3

Given the following information for accident year 2012 as of December 31, 2012:

- $\diamond$ $U_0 = \$5{,}000$
- $\diamond$ $C_k = \$3{,}000$
- $\diamond$ $q_k = 0.60$

a) Calculate $U^{(3)}$.

b) Calculate $U^{(\infty)}$.

Solution to part a:

- ◇ $U^{(1)} = U_{BF} = C_k + q_k U_0 = 3000 + 0.6(5000) = 6000$
- ◇ $U^{(2)} = U_{GB} = C_k + q_k U_{BF} = 3000 + 0.6(6000) = 6600$
- ◇ $U^{(3)} = C_k + q_k U_{GB} = 3000 + 0.6(6600) = \boxed{\$6{,}960}$

Solution to part b:

- ◇ $U^{(\infty)} = U_{CL} = C_k / p_k = 3000/(1 - 0.6) = \boxed{\$7{,}500}$

MP #4

Given the following information for accident year 2012 as of December 31, 2012:

- ◇ 12-ultimate cumulative paid LDF = 2.50
- ◇ Reserve based on the chain-ladder method = $4,000
- ◇ Ultimate loss based on the Benktander method = $8,000

Using a credibility weight of $c = p_k$, calculate the accident year 2012 Esa Hovinen reserve.

Solution:

- ◇ When $c = p_k$, $R_{EH} = R_{GB} = U_{GB} - C_k$
- ◇ To determine C_k:
  - $R_{CL} = q_k U_{CL}$
  - $U_{CL} = 4000/(1 - \frac{1}{2.5}) = 6666.667$
  - Thus, $C_k = U_{CL} - R_{CL} = 6666.667 - 4000 = 2666.667$
- ◇ Plugging C_k into our formula for R_{EH}, we find that $R_{EH} = 8000 - 2666.667 = \boxed{\$5{,}333.33}$

MP #5

Given the following information for accident year 2012 as of December 31, 2012:

- ◇ $c^* = 0.32$
- ◇ $C_k = \$3,000$
- ◇ $U_{CL} = \$5,000$

Which reserve has a smaller MSE: R_{GB} or R_{BF}?

Solution:

- ◇ $U_{CL} = C_k/p_k$. Thus, $p_k = 0.6$
- ◇ If $c^* > p_k/2$, R_{GB} has a smaller MSE
- ◇ Checking the condition above, $0.32 > 0.6/2$
- ◇ Thus, $\boxed{R_{GB} \text{ has a smaller MSE}}$

Past CAS Exam Problems & Solutions

2018 #5

Given the following information about accident year 2017 as of December 31, 2017:

- ◇ Accident year 2017 paid loss = \$850,000
- ◇ 2017 earned premium = \$4,000,000
- ◇ Initial expected loss ratio = 67.5%
- ◇ 12-24 month incremental paid link ratio = 1.60
- ◇ 12-ultimate cumulative paid LDF = 3.00

a) Determine the accident year 2017 incremental paid loss in 2018 that would result in the Benktander ultimate loss estimate being \$100,000 less than the Bornhuetter-Ferguson ultimate loss estimate for accident year 2017 as of December 31, 2018. Assume all development factors are unchanged.

b) Briefly describe when the Benktander ultimate loss estimate would be greater than the Bornhuetter-Ferguson ultimate loss estimate as of December 31, 2018.

c) Explain why it may not be appropriate to use the Bornhuetter-Ferguson method when losses develop downward.

Solution to part a:

- ◇ $U_{BF} = C_K + U_0 q_k = (850 + x) + 4000(0.675)\left(1 - \frac{1}{3/1.6}\right) = 2110 + x$. *Notice here that we are dividing 3 by 1.6 to obtain the cumulative paid LDF at 24 months*
- ◇ $U_{GB} = C_k + U_{BF} q_k = (850 + x) + (2110 + x)\left(1 - \frac{1}{3/1.6}\right)$. Since we want U_{GB} to be 100,000 less than U_{BF}, we have $(850 + x) + (2110 + x)\left(1 - \frac{1}{3/1.6}\right) = 2110 + x - 100$. Thus, $x = \boxed{\$375{,}714}$

Solution to part b:

- ◇ Since the Benktander estimate is a weighting of the CL estimate and the BF estimate, the Benktander estimate is greater than the BF estimate when the CL estimate is greater than the BF estimate

Solution to part c:

- ◇ Since the BF IBNR does not respond to actual loss performance, the downward development will not affect IBNR produced by the BF method. If the downward development represents real trends (such as increased salvage and subrogation), then the BF method will overstate the IBNR

2013 #4

Given the following information:

	Cumulative Paid Loss ($000)			
AY	12 mo.	24 mo.	36 mo.	48 mo.
2009	5,751	10,640	11,491	12,181
2010	5,528	9,287	10,680	
2011	4,120	7,004		
2012	5,304			

	Calculated Ultimate Loss ($000)	
Accident Year	Bornhuetter/Ferguson Ultimate	Benktander Ultimate
2009	12,181	12,181
2010	11,246	11,316
2011	8,428	8,204
2012	10,403	10,609

a) Calculate the 24-month-to-ultimate cumulative development factor that would result in the ultimate loss estimates shown above.

b) For accident year 2011, suppose that the Bornhuetter/Ferguson method is performed over multiple iterations. Deduce the ultimate loss estimate that will be produced as the number of iterations approaches infinity.

Solution to part a:

- ◇ Since we want to calculate the 24-ultimate development factor, let's look at AY 2011
- ◇ $U_{GB} = C_k + q_k U_{BF}$
- ◇ $8204 = 7004 + q_k(8428)$
- ◇ $q_k = 0.142$
- ◇ $0.142 = 1 - \frac{1}{LDF_{24-ult}}$
- ◇ Thus, $LDF_{24-ult} = \boxed{1.166}$

Solution to part b:

- ◇ As the number of Bornhuetter/Ferguson iterations approaches infinity, the chain-ladder ultimate loss estimate will be produced

2012 #1

Given the following information for accident year 2011 as of December 31, 2011:

- ◇ Accident year 2011 paid loss = \$700,000
- ◇ 2011 earned premium = \$3,000,000
- ◇ Initial expected loss ratio = 62.5%
- ◇ 12-24 month paid link ratio = 1.50
- ◇ 12-ultimate cumulative paid LDF = 2.50

a) Calculate accident year 2011 ultimate loss estimates as of December 31, 2011 using each of the following three methods:

  - ◇ Chain ladder
  - ◇ Bornhuetter/Ferguson
  - ◇ Benktander

b) Determine the accident year 2011 incremental paid loss in 2012 that would result in the Benktander ultimate loss estimate being \$50,000 greater than the Bornhuetter/Ferguson ultimate loss estimate for accident year 2011, as of December 31, 2012. Assume all selected development factors remain the same.

Solution to part a:

◇ Chain-ladder

- $U_{CL} = 700000(2.5) = \boxed{\$1{,}750{,}000}$

◇ Bornhuetter/Ferguson

- $U_{BF} = C_k + q_k U_0 = 700000 + (1 - 1/2.5)(3000000)(0.625) = \boxed{\$1{,}825{,}000}$

◇ Benktander

- $U_{GB} = C_k + q_k U_{BF} = 7000000 + (1 - 1/2.5)(1825000) = \boxed{\$1{,}795{,}000}$

Solution to part b:

◇ $U_{GB} = U_{BF} + 50000$

◇ $C_k + q_k U_{BF} = U_{BF} + 50000$

◇ $C_k - 50000 = U_{BF}(1 - q_k)$

◇ Let the incremental paid loss in 2012 for AY 2011 be x

◇ $700000 + x - 50000 = U_{BF}(1 - q_k)$

◇ $650000 + x = U_{BF}(p_k)$

◇ $650000 + x = U_{BF}\left(\frac{1}{LDF_{24-ult}}\right)$

◇ $650000 + x = U_{BF}\left(\frac{1}{2.5/1.5}\right)$

◇ $650000 + x = U_{BF}(0.6)$

◇ $650000 + x = (C_k + q_k U_0)(0.6)$

◇ $650000 + x = (700000 + x + 0.4(3000000)(0.625))(0.6)$

◇ $650000 + x = 870000 + 0.6x$

◇ $0.4x = 220000$

◇ $x = \boxed{\$550{,}000}$

Hürlimann

Outline

I. Introduction

◇ Hürlimann's method is inspired by the Benktander method

◇ A couple of differences between Hürlimann's method and the Benktander method:

- Hürlimann's method is based on a full development triangle, whereas the Benktander method is based on a single origin period (i.e. accident year or underwriting year)
- Hürlimann's method requires a measure of exposure for each origin period (i.e. premiums)

◇ Unlike standard reserving methods that rely on link ratios to determine reserves (chain-ladder, Bornhuetter/Ferguson, Cape Cod), Hürlimann's method relies on loss ratios

◇ The **main result of the method** is that it provides an optimal credibility weight for combining the chain-ladder or individual loss ratio reserve (grossed up latest claims experience of an origin period) with the Bornhuetter/Ferguson or collective loss ratio reserve (experience based burning cost estimate of the total ultimate claims of an origin period)

II. The Collective and Individual Loss Ratio Claims Reserves

◇ Notation

- p_i is the proportion of the total ultimate claims from origin period i expected to be paid in development period $n - i + 1$ (known as the loss ratio payout factor or loss ratio lag-factor)
- $q_i = 1 - p_i$ is the proportion of the total ultimate claims from origin period i which remain unpaid in development period $n - i + 1$ (known as the loss ratio reserve factor)
- $U_i^{BC} = U_i^{(0)}$ is the burning cost of the total ultimate claims for origin period i
- $U_i^{coll} = U_i^{(1)}$ is the collective total ultimate claims for origin period i
- $U_i^{ind} = U_i^{(\infty)}$ is the individual total ultimate claims for origin period i
- $U_i^{(m)}$ is the ultimate claim estimate at the m^{th} iteration for origin period i
- R_i^{coll} is the collective loss ratio claims reserve for origin period i
- R_i^{ind} is the individual loss ratio claims reserve for origin period i

- R_i^c is the credible loss ratio claims reserve
- R_i^{GB} is the Benktander loss ratio claims reserve
- R_i^{WN} is the Neuhaus loss ratio claims reserve
- R_i is the i-th period claims reserve for origin period i
- R is the total claims reserve
- m_k is the expected loss ratio in development period k
- n is the number of origin periods
- V_i is the premium belonging to origin period i
- S_{ik} are the paid claims from origin period i as of k years of development where $1 \leq i, k \leq n$
- C_{ik} are the cumulative paid claims from origin period i as of k years of development

◇ Assuming that after n development periods all claims incurred in an origin period are known and closed, the **total ultimate claims** from origin period i are:

$$\boxed{\sum_{k=1}^{n} S_{ik}}$$

◇ **Cumulative paid claims**

$$\boxed{C_{ik} = \sum_{j=1}^{k} S_{ij}}$$

◇ ***i*-th period claims reserve**

- The required amount for the incurred but unpaid claims of origin period i

$$\boxed{R_i = \sum_{k=n-i+2}^{n} S_{ik}}$$

where $i = 2, ..., n$

◇ **Total claims reserve**

- The total amount of incurred but unpaid claims over all periods

$$\boxed{R = \sum_{i=2}^{n} R_i}$$

◇ **Expected loss ratio**

- The incremental amount of expected paid claims per unit of premium in each development period (i.e. an incremental loss ratio)

$$\boxed{m_k = \frac{E\left[\sum\limits_{i=1}^{n-k+1} S_{ik}\right]}{\sum\limits_{i=1}^{n-k+1} V_i}}$$

where $k = 1, ..., n$

◇ **Expected value of the burning cost** of the total ultimate claims

- This quantity is similar to the prior estimate U_0 from Mack (2000)

$$\boxed{E\left[U_i^{BC}\right] = V_i \cdot \sum_{k=1}^{n} m_k}$$

- By summing up the m_k's (the incremental loss ratios), we obtain an overall expected loss ratio. When we multiply the overall expected loss ratio by the premium V_i, we obtain an expected loss for each origin period

◇ **Loss ratio payout factor**

- Represents the percent of losses emerged to date for each origin period

$$\boxed{\begin{aligned} p_i &= \frac{V_i \cdot \sum\limits_{k=1}^{n-i+1} m_k}{E[U_i^{BC}]} \\ &= \frac{\sum\limits_{k=1}^{n-i+1} m_k}{\sum\limits_{k=1}^{n} m_k} \end{aligned}}$$

◇ **Individual total ultimate claims**

- Obtained by grossing up the latest cumulative paid claims for an origin period
- Considered "individual" since it depends on the individual latest claims experience of an origin period

- This estimate is similar to the chain-ladder (CL) estimate from Mack (2000)

$$\boxed{U_i^{ind} = \frac{C_{i,n-i+1}}{p_i}}$$

◇ **Individual loss ratio claims reserve**

$$\boxed{\begin{aligned} R_i^{ind} &= U_i^{ind} - C_{i,n-i+1} \\ &= q_i \cdot U_i^{ind} \\ &= \frac{q_i}{p_i} \cdot C_{i,n-i+1} \end{aligned}}$$

◇ **Collective loss ratio claims reserve**

- Obtained by using the burning cost of the total ultimate claims
- Considered "collective" since it depends on the portfolio claims experience of all origin periods

$$\boxed{R_i^{coll} = q_i \cdot U_i^{BC}}$$

◇ **Collective total ultimate claims**

- This estimate is similar to the Bornhuetter/Ferguson (BF) estimate from Mack (2000)

$$\boxed{U_i^{coll} = R_i^{coll} + C_{i,n-i+1}}$$

◇ An **advantage** of the collective loss ratio claims reserve over the BF reserve is that different actuaries always come to the same results provided they use the same premiums

III. Credible Loss Ratio Claims Reserve

◇ The individual and collective loss ratio claims reserve estimates represent extreme positions

- The individual claims reserve assumes that the cumulative paid claims amount $C_{i,n-i+1}$ is fully credible for future claims and ignores the burning cost U_i^{BC} of the total ultimate claims
- The collective claims reserve ignores the cumulative paid claims and relies fully on the burning cost

◇ **Credible loss ratio claims reserve**

- Mixture of the individual and collective loss ratio reserves

$$\boxed{R_i^c = Z_i \cdot R_i^{ind} + (1 - Z_i) \cdot R_i^{coll}}$$

where Z_i is the credibility weight given to the individual loss ratio reserve

◇ **Benktander loss ratio claims reserve**

- Obtained by setting $Z_i = Z_i^{GB} = p_i$

$$\boxed{R_i^{GB} = p_i \cdot R_i^{ind} + q_i \cdot R_i^{coll}}$$

◇ **Neuhaus loss ratio claims reserve**

- Obtained by setting $Z_i = Z_i^{WN} = \sum_{k=1}^{n-i+1} m_k = p_i \cdot \sum_{k=1}^{n} m_k$

$$\boxed{R_i^{WN} = Z_i^{WN} \cdot R_i^{ind} + (1 - Z_i^{WN}) \cdot R_i^{coll}}$$

◇ At this point in the paper, Hürlimann restates the theorem from Mack (2000) that shows how ultimates and reserves change as we iterate between them

◇ Using the iteration rules $R_i^{(m)} = q_i U_i^{(m)}$ and $U_i^{(m+1)} = C_{i,n-i+1} + q_i U_i^{(m)}$, we obtain the following credibility mixtures:

$$\boxed{\begin{aligned} U_i^{(m)} &= (1 - q_i^m)U_i^{ind} + q_i^m U_i^0 \\ R_i^{(m)} &= (1 - q_i^m)R_i^{ind} + q_i^m R_i^0 \end{aligned}}$$

◇ Once again, if we iterate between reserves and ultimates indefinitely, we eventually end up with the individual loss ratio estimate for ultimate claims.

IV. The Optimal Credibility Weights and the Mean Squared Error

◇ The optimal credibility weights Z_i^* which minimize the mean squared error $\text{mse}(R_i^c) = E[(R_i^c - R_i)^2]$ are given by:

$$\boxed{Z_i^* = \frac{p_i}{p_i + t_i}}$$

where $t_i = \frac{E[\alpha_i^2(U_i)]}{Var(U_i^{BC}) + Var(U_i) - E[\alpha_i^2(U_i)]}$

◇ *In the paper, the author goes into quite a bit of detail on how to estimate the quantities in the formula for t_i above. I believe that these details are outside of the scope of the exam and are excluded from this outline*

◇ The weights Z_i^* which minimize the mean squared error $\text{mse}(R_i^c) = E[(R_i^c - R_i)^2]$ and the variance $Var(R_i^c)$ are obtained by:

$$\boxed{t_i^* = \frac{f_i - 1 + \sqrt{(f_i + 1) \cdot (f_i - 1 + 2p_i)}}{2}}$$

◇ *Note that f_i comes from an assumption the author makes in the paper. He assumes that U_i is at least as volatile as the burning cost estimate U_i^{BC}. Thus, $Var(U_i) = f_i \cdot Var(U_i^{BC})$*

- A special case of the formula above is when $f_i = 1$. This implies that $Var(U_i) = Var(U_i^{BC})$. In this case, t_i can be estimated by

$$\boxed{t_i^* = \sqrt{p_i}}$$

  This is the case I expect to see on the exam. Thus, unless told otherwise, assume that $t_i = t_i^* = \sqrt{p_i}$. *Note that the online CAS text references provide two different versions of this paper. Each version of the paper has a different version of the formula above. If you navigate to the online text references and click on the first link under Hürlimann, you will find that $t_i^* = \sqrt{p_i}$. If you download the "complete PDF of online text references," it provides the second version of this paper with a different formula for t_i^*. Given that $t_i^* = \sqrt{p_i}$ is what is shown in all of the solutions on prior exams, I recommend using this version of the formula*

- Since $t_i^* = \sqrt{p_i} \leq 1$, $Z_i^* \leq \frac{1}{2}$

- **According to the author**, this special case is appealing because it yields the smallest credibility weights for the individual loss reserves, which places more emphasis on the collective loss reserves (*I say "According to the author" because this is not correct. As f increases from $f = 1$, the credibility Z actually decreases, placing less weight on the individual loss reserves. If this comes up as a short answer question on the exam, stick with what the author says*)

- The mean squared error for the credible loss ratio reserve is given by:

$$\boxed{\text{mse}(R_i^c) = E[\alpha_i^2(U_i)] \cdot \left[\frac{Z_i^2}{p_i} + \frac{1}{q_i} + \frac{(1-Z_i)^2}{t_i}\right] \cdot q_i^2}$$

- The mean squared errors for the collective and individual loss ratios reserves can be obtained by setting Z_i equal to 0 and 1, respectively

V. Example

- Given the following incremental losses:

		Dev. Period		
i	V_i = Premium	1	2	3
1	15	10	4	2
2	20	6	5	
3	22	8		

◇ Calculate the following parameters:

i or k	m_k	$p_i = Z_i^{GB}$	q_i	t_i^*	Z_i^*	Z_i^{WN}
1	0.421	1.000	0.000	1.000	0.500	0.811
2	0.257	0.836	0.164	0.914	0.478	0.678
3	0.133	0.519	0.481	0.720	0.419	0.421

◇ Here are the underlying calculations:

- $m_k = \frac{E\left[\sum_{i=1}^{n-k+1} S_{ik}\right]}{\sum_{i=1}^{n-k+1} V_i}$
  - ◇ $m_1 = \frac{10+6+8}{15+20+22} = 0.421$
  - ◇ $m_2 = \frac{4+5}{15+20} = 0.257$
  - ◇ $m_3 = \frac{2}{15} = 0.133$
- $p_i = \frac{\sum_{k=1}^{n-i+1} m_k}{\sum_{k=1}^{n} m_k}$
  - ◇ $p_1 = \frac{0.421+0.257+0.133}{0.421+0.257+0.133} = 1.000$
  - ◇ $p_2 = \frac{0.421+0.257}{0.421+0.257+0.133} = 0.836$
  - ◇ $p_3 = \frac{0.421}{0.421+0.257+0.133} = 0.519$
- $q_i = 1 - p_i$
  - ◇ $q_1 = 1 - 1 = 0.000$
  - ◇ $q_2 = 1 - 0.836 = 0.164$
  - ◇ $q_3 = 1 - 0.519 = 0.481$
- $t_i^* = \sqrt{p_i}$ (assumes that $Var(U_i) = Var(U_i^{BC})$)
  - ◇ $t_1^* = \sqrt{1} = 1.000$
  - ◇ $t_2^* = \sqrt{0.836} = 0.914$
  - ◇ $t_3^* = \sqrt{0.519} = 0.720$
- $Z_i^* = \frac{p_i}{p_i + t_i^*}$
  - ◇ $Z_1^* = \frac{1}{1+1} = 0.500$
  - ◇ $Z_2^* = \frac{0.836}{0.836+0.914} = 0.478$
  - ◇ $Z_3^* = \frac{0.519}{0.519+0.720} = 0.419$

- $Z_i^{WN} = \sum_{k=1}^{n-i+1} m_k$
  - ◇ $Z_1^{WN} = 0.421 + 0.257 + 0.133 = 0.811$
  - ◇ $Z_2^{WN} = 0.421 + 0.257 = 0.678$
  - ◇ $Z_3^{WN} = 0.421$

◇ Calculate the reserves:

i	Collective	Individual	Neuhaus	Benktander	Optimal
2	2.660	2.158	2.320	2.240	2.420
3	8.582	7.414	8.090	7.976	8.093

◇ Here are the underlying calculations for the collective, individual, and Neuhaus reserves for origin period 2:

- Collective $= q_i \cdot U_i^{BC} = 0.164(20)(0.421 + 0.257 + 0.133) = 2.660$ (similar to BF)
- Individual $= \frac{C_{i,n-i+1}}{p_i} - C_{i,n-i+1} = \frac{6+5}{0.836} - (6+5) = 2.158$ (similar to CL)
- Neuhaus $= Z_i^{WN} \cdot R_i^{ind} + (1 - Z_i^{WN}) \cdot R_i^{coll} = 0.678(2.158) + (1 - 0.678)(2.660) = 2.320$

◇ Calculate the relative MSE's for each method (i.e. divide each method's MSE by the optimal MSE):

i	Collective	Individual	Neuhaus	Benktander	Optimal
2	1.078	1.094	1.014	1.044	1.000
3	1.202	1.388	1.000	1.012	1.000

◇ Here are the underlying calculations for the collective, individual, and Neuhaus reserves for origin period 2:

- Collective $= \dfrac{E[\alpha_i^2(U_i)] \cdot \left[\frac{0^2}{0.836} + \frac{1}{0.164} + \frac{(1-0)^2}{0.914}\right] \cdot 0.164^2}{E[\alpha_i^2(U_i)] \cdot \left[\frac{0.478^2}{0.836} + \frac{1}{0.164} + \frac{(1-0.478)^2}{0.914}\right] \cdot 0.164^2} = 1.078$
- Individual $= \dfrac{E[\alpha_i^2(U_i)] \cdot \left[\frac{1^2}{0.836} + \frac{1}{0.164} + \frac{(1-1)^2}{0.914}\right] \cdot 0.164^2}{E[\alpha_i^2(U_i)] \cdot \left[\frac{0.478^2}{0.836} + \frac{1}{0.164} + \frac{(1-0.478)^2}{0.914}\right] \cdot 0.164^2} = 1.094$
- Neuhaus $= \dfrac{E[\alpha_i^2(U_i)] \cdot \left[\frac{0.678^2}{0.836} + \frac{1}{0.164} + \frac{(1-0.678)^2}{0.914}\right] \cdot 0.164^2}{E[\alpha_i^2(U_i)] \cdot \left[\frac{0.478^2}{0.836} + \frac{1}{0.164} + \frac{(1-0.478)^2}{0.914}\right] \cdot 0.164^2} = 1.014$

◇ Using the relative MSE table, it's clear that the Neuhaus reserve best matches the optimal credible reserve

VI. Reinterpreting the Methods from Mack (2000)

- *Note: In this section, the author is making connections between this paper and the Mack (2000) paper. Thus, we are using the standard age-to-age factors in this section*
- Let $f_k^{CL} = \frac{\sum_{i=1}^{n-k} C_{i,k+1}}{\sum_{i=1}^{n-k} C_{ik}}$. These are the chain-ladder age-to-age factors
- Let $F_k^{CL} = \prod_{j=k}^{n-1} f_j^{CL}$. These are the chain-ladder age-to-ultimate factors
- Let $p_i^{CL} = \frac{1}{F_{n-i+1}^{CL}}$. These are the chain-ladder lag-factors
- Let $q_i^{CL} = 1 - p_i^{CL}$. These are the chain-ladder reserve factors
- **Chain-ladder method**
  - This is the individual loss ratio method with loss ratio lag-factors replaced by the chain-ladder lag-factors:

$$\boxed{R_i^{CL} = \frac{q_i^{CL}}{p_i^{CL}} \cdot C_{i,n-i+1}}$$

- **Cape Cod method**
  - Benktander-type credibility mixture with the following components:

$$\boxed{\begin{aligned} R_i^{\text{ind}} &= \frac{q_i^{CL}}{p_i^{CL}} \cdot C_{i,n-i+1} \\ R_i^{\text{coll}} &= q_i^{CL} \cdot LR \cdot V_i \\ Z_i &= p_i^{CL} \end{aligned}}$$

  where $LR = \frac{\sum_{i=1}^{n} C_{i,n-i+1}}{\sum_{i=1}^{n} p_i^{CL} \cdot V_i}$

  - *Note: The credibility mixture above does not equal the Cape Cod method. Instead, the collective reserves defined above equal the standard Cape Cod reserves. Thus, the credibility estimate is mixture of the chain-ladder reserve estimate and the standard Cape Cod reserve estimate*
- **Optimal Cape Cod method**
  - Identical to the Cape Cod method, but with the following credibility weights:

$$\boxed{Z_i = \frac{p_i^{CL}}{p_i^{CL} + \sqrt{p_i^{CL}}}}$$

◇ **Bornhuetter/Ferguson method**

- Benktander-type credibility mixture with the following components:

$$\boxed{\begin{aligned} R_i^{\text{ind}} &= \frac{q_i^{CL}}{p_i^{CL}} \cdot C_{i,n-i+1} \\ R_i^{\text{coll}} &= q_i^{CL} \cdot LR_i \cdot V_i \\ Z_i &= p_i^{CL} \end{aligned}}$$

  where LR_i is some selected initial loss ratio for each origin period

- *Note: The credibility mixture above does not equal the BF method. Instead, the collective reserves defined above equal the standard BF reserves. Thus, the credibility estimate is mixture of the chain-ladder reserve estimate and the standard BF reserve estimate*

◇ **Optimal Bornhuetter/Ferguson method**

- Identical to the Bornhuetter/Ferguson method, but with the following credibility weights:

$$\boxed{Z_i = \frac{p_i^{CL}}{p_i^{CL} + \sqrt{p_i^{CL}}}}$$

Clark

Outline

I. Introduction

◇ Objectives in creating a formal model of loss reserving:

- Describe loss emergence in simple mathematical terms as a guide to selecting amounts for carried reserves
- Provide a means of estimating the range of possible outcomes around the "expected" reserve

◇ A statistical loss reserving model has two key elements:

- The expected amount of loss to emerge in some time period
- The distribution of actual emergence around the expected value

II. Expected Loss Emergence

◇ Model will estimate the expected amount of loss to emerge based on:

- An estimate of the ultimate loss by year
- An estimate of the pattern of loss emergence

◇ Let $G(x) = 1/LDF_x$ be the cumulative % of loss reported (or paid) as of time x, where x represents the time (in months) from the "average" accident date to the evaluation date

◇ Assume that the loss emergence pattern is described by one of the following curves with scale θ and shape ω

- Loglogistic

$$\boxed{\begin{aligned} G(x|\omega,\theta) &= \frac{x^\omega}{x^\omega + \theta^\omega} \\ LDF_x &= 1 + \theta^\omega \cdot x^{-\omega} \end{aligned}}$$

- Weibull

$$\boxed{G(x|\omega,\theta) = 1 - \exp(-(x/\theta)^\omega)}$$

◇ With these curves, we assume a strictly increasing pattern. If there is real expected negative development (salvage recoveries), different models should be used

◇ Advantages to using parameterized curves to describe the emergence pattern:

- Estimation is simple since we only have to estimate two parameters
- We can use data that is not from a triangle with evenly spaced evaluation data – such as the case in which the latest diagonal is only nine months from the second latest diagonal
- The final pattern is smooth and does not follow random movements in the historical age-to-age factors

◇ In order to estimate the loss emergence amount, we require an estimate of the ultimate loss by AY. There are two methods described in the paper:

- LDF method – assumes the loss amount in each AY is independent from all other years (this is the standard chain-ladder method)
- Cape Cod method – assumes that there is a known relationship between expected ultimate losses across accident years, where the relationship is identified by an exposure base (on-level premium, sales, payroll, etc.)

◇ Let $\mu_{AY;x,y}$ = expected incremental loss dollars in accident year AY between ages x and y

◇ Combining the loss emergence pattern with the estimate of the ultimate loss by year, we obtain the following for each method:

- LDF method

$$\boxed{\mu_{AY;x,y} = ULT_{AY} \cdot [G(y|\omega, \theta) - G(x|\omega, \theta)]}$$

- Cape Cod method

$$\boxed{\mu_{AY;x,y} = \text{Premium}_{AY} \cdot ELR \cdot [G(y|\omega, \theta) - G(x|\omega, \theta)]}$$

◇ In general, the Cape Cod method is preferred since data is summarized into a loss triangle with relatively few data points. Since the LDF method requires an estimation of a number of parameters (one for each AY ultimate loss, as well as θ and ω), it tends to be over-parameterized when few data points exist

◇ Due to the additional information given by the exposure base (as well as fewer parameters), the Cape Cod method has a smaller parameter variance. The process variance can be higher or lower than the LDF method. In general, the Cape Cod method produces a lower total variance than the LDF method

III. The Distribution of Actual Loss Emergence and Maximum Likelihood

◇ The variance of the actual loss emergence can be estimated in two pieces: **process variance** (the "random" amount) and **parameter variance** (the uncertainty in the estimator, also known as the estimation error)

◇ **Process variance**

- Assume that the loss in any period has a constant ratio of variance/mean:

$$\boxed{\frac{\text{Variance}}{\text{Mean}} = \sigma^2 \approx \frac{1}{n-p}\sum_{AY,t}^{n} \frac{(c_{AY,t} - \mu_{AY,t})^2}{\mu_{AY,t}}}$$

  where n = # of data points, p = # of parameters, $c_{AY,t}$ = actual incremental loss emergence and $\mu_{AY,t}$ = expected incremental loss emergence

- For estimating the parameters of our model, let's assume that the actual loss emergence "c" follows an over-dispersed Poisson distribution with scaling factor σ^2

- Assuming λ represents the mean of a standard Poisson random variable, the mean and variance of an over-dispersed Poisson are as follows:

  ◇ $E[c] = \lambda\sigma^2 = \mu$

  ◇ $Var(c) = \lambda\sigma^4 = \mu\sigma^2$

- Key advantages of using the over-dispersed Poisson distribution:

  ◇ Inclusion of scaling factors allows us to match the first and second moments of any distribution, allowing high flexibility

  ◇ Maximum likelihood estimation produces the LDF and Cape Cod estimates of ultimate losses, so the results can be presented in a familiar format

◇ **The likelihood function**

- For an over-dispersed Poisson distribution, the $\Pr(c) = \frac{\lambda^{c/\sigma^2} e^{-\lambda}}{(c/\sigma^2)!}$

- Likelihood $= \prod_i \Pr(c_i) = \prod_i \frac{\lambda_i^{c_i/\sigma^2} e^{-\lambda_i}}{(c_i/\sigma^2)!} = \prod_i \frac{(\mu_i/\sigma^2)^{c_i/\sigma^2} e^{-(\mu_i/\sigma^2)}}{(c_i/\sigma^2)!}$

- After taking the log of the likelihood function above, we obtain the loglikelihood, l, which we need to maximize:

$$\boxed{l = \sum_i c_i \cdot \ln(\mu_i) - \mu_i}$$

- Before applying this loglikelihood formula to our two methods, let's define a few things:

  ◇ $c_{i,t}$ = actual loss in AY i, development period t

  ◇ P_i = premium for AY i

  ◇ x_{t-1} = beginning age for development period t

  ◇ x_t = ending age for development period t

- LDF method

- ◇ Taking the derivative of l and setting it equal to zero yields the following MLE estimate for ULT_i:

$$\boxed{ULT_i = \frac{\sum_t c_{i,t}}{\sum_t [G(x_t) - G(x_{t-1})]}}$$

- ◇ The MLE estimate for each ULT_i is equivalent to the "LDF Ultimate"

- Cape Cod method

  - ◇ Taking the derivative of l and setting it equal to zero yields the following MLE estimate for the ELR:

$$\boxed{ELR = \frac{\sum_{i,t} c_{i,t}}{\sum_{i,t} P_i \cdot [G(x_t) - G(x_{t-1})]}}$$

  - ◇ The MLE estimate for the ELR is equivalent to the "Cape Cod" Ultimate

- An **advantage** of the maximum loglikelihood function is that it works in the presence of negative or zero incremental losses (since we never actually take the log of $c_{i,t}$)

◇ **Parameter variance**

- We need the covariance matrix (inverse of the information matrix) to calculate the parameter variance
- Due to the complexity involved (it would be downright impossible for the LDF method), I don't expect you will need to calculate the parameter variance on the exam

◇ **Variance of the reserves**

- As usual, in order to calculate the variance of an estimate of loss reserves R, we need the process variance and parameter variance:
  - ◇ Process Variance of $R = \sigma^2 \cdot \sum \mu_{AY;x,y}$
  - ◇ Parameter Variance of $R =$ too complicated for the exam

IV. Key Assumptions of this Model

◇ Assumption 1: Incremental losses are independent and identically distributed (iid)

- "Independence" means that one period does not affect the surrounding periods
  - ◇ Can be tested using residual analysis
  - ◇ Positive correlation could exist if all periods are equally impacted by a change in loss inflation

    - ◇ Negative correlation could exist if a large settlement in one period replaces a stream of payments in later periods
  - "Identically distributed" assumes that the emergence pattern is the same for all accident years, which is clearly over-simplified
    - ◇ Different risks and a different mix of business would have been written in each historical period, each subject to different claims handling and settlement practices

◇ Assumption 2: The variance/mean scale parameter σ^2 is fixed and known

  - Technically, σ^2 should be estimated simultaneously with the other model parameters, with the variance around its estimate included in the covariance matrix
  - However, doing so results in messy mathematics. For convenience and simplicity, we assume that σ^2 is fixed and known

◇ Assumption 3: Variance estimates are based on an approximation to the Rao-Cramer lower bound

  - The estimate of variance based on the information matrix is only exact when we are using linear functions
  - Since our model is non-linear, the variance estimate is a Rao-Cramer lower bound (i.e. the variance estimate is as low as it possibly can be)

V. A Practical Example

◇ *In the paper, Clark applies his methodology to 10 x 10 triangle. To simplify things, we will be studying a 5 x 5 triangle. In general, this example will focus on estimating the reserves using the LDF and Cape Cod methods. For the more detailed calculations (such as determining model parameters or calculating residuals), see the Clark Example excel spreadsheet within the online course.*

◇ Before diving into the example, let's briefly discuss growth curve extrapolation:

- The growth curve extrapolates reported losses to ultimate
- For curves with "heavy" tails (such as loglogistic), it may be necessary to truncate the LDF at a finite point in time to reduce reliance on the extrapolation
- An alternative to truncating the tail factor is using a growth curve with a "lighter" tail (such as Weibull)

◇ **LDF method**

- Assume that expected loss emergence is described by a loglogistic curve. In addition, assume that the curve should be truncated at 120 months
- Given the following cumulative losses and parameters:

	Cumulative Losses ($)				
AY	12	24	36	48	60
2010	500	1500	2250	2590	2720
2011	550	1700	2400	2725	
2012	450	1200	2000		
2013	600	1750			
2014	575				

Parameters	
θ	21.4675
ω	1.477251
σ^2	59.9876

- Create the following table to estimate the reserves:

AY	Losses at 12/31/14	Age at 12/31/14	Avg. Age (x)	Growth Function	Fitted LDF	Trunc. LDF	Estimated Reserves	Estimated Ultimate
Trunc.		120	114	0.922	1.0846	1.0000		
2010	2720	60	54	0.796	1.2563	1.1583	430.576	3150.576
2011	2725	48	42	0.729	1.3717	1.2647	721.308	3446.308
2012	2000	36	30	0.621	1.6103	1.4847	969.400	2969.400
2013	1750	24	18	0.435	2.2989	2.1195	1959.125	3709.125
2014	575	12	6	0.132	7.5758	6.9848	3441.260	4016.260
Total							7521.669	17291.669

- Here are the 2013 calculations for the table above:
  - ◇ Avg. age = 18 = 24 − 6
  - ◇ Growth function $= \frac{x^\omega}{x^\omega+\theta^\omega} = \frac{18^{1.477251}}{18^{1.477251}+21.4675^{1.477251}} = 0.435$
  - ◇ Fitted LDF $= \frac{1}{0.435} = 2.2989$
  - ◇ Truncated LDF $= \frac{0.922}{0.435} = 2.1195$
  - ◇ Estimated reserves $= 1750(2.1195 - 1) = 1959.125$
  - ◇ Estimated ultimate $= 1750 + 1959.125 = 3709.125$
- To calculate the process standard deviations of the reserves for each accident year, we multiply the scale parameter σ^2 by the estimated reserves and take the square root. Thus, we have the following:

AY	Estimated Reserves	Process SD
2010	430.576	160.715
2011	721.308	208.013
2012	969.400	241.147
2013	1959.125	$342.817 = \sqrt{59.9876(1959.125)}$
2014	3441.260	454.349
Total	7521.669	671.719

◇ **CC method**

- Assume that expected loss emergence is described by a Loglogistic curve. In addition, assume that the curve should be truncated at 120 months
- Given the following cumulative loss and parameters:

	Cumulative Losses ($)				
AY	12	24	36	48	60
10	500	1500	2250	2590	2720
11	550	1700	2400	2725	
12	450	1200	2000		
13	600	1750			
14	575				

Parameters	
θ	22.3671
ω	1.441024
σ^2	50.0730

- Create the following table to calculate the ELR (**note that the ELR is calculated before truncation to remain algebraically consistent with how the LDF method works**):

AY	On-Level Premium	Losses at 12/31/14	Age at 12/31/14	Avg. Age (x)	Growth Function	Premium × Growth
2010	5000	2720	60	54	0.781	3905.00
2011	5200	2725	48	42	0.713	3707.60
2012	5400	2000	36	30	0.604	3261.60
2013	5600	1750	24	18	0.422	2363.20
2014	5800	575	12	6	0.131	759.80

- Here are the 2013 calculations for the table above:
  - ◇ Average age = 18 = 24 − 6
  - ◇ Growth function $= \frac{x^\omega}{x^\omega+\theta^\omega} = \frac{18^{1.441024}}{18^{1.441024}+22.3671^{1.441024}} = 0.422$
  - ◇ Premium × growth = 5600(0.422) = 2363.20
- The expected loss ratio is $\frac{2720+2725+2000+1750+575}{3905+3707.60+3261.60+2363.20+759.80} = 0.698$
- Assuming a truncation point of 120 months, estimate the reserves:

AY	On-Level Premium	Age at 12/31/14	Average Age (x)	Growth Function	0.913 − Growth	Expected Losses	Estimated Reserves
Trunc.		120	114	0.913	0.000		
2010	5000	60	54	0.781	0.132	3490.00	460.680
2011	5200	48	42	0.713	0.200	3629.60	725.920
2012	5400	36	30	0.604	0.309	3769.20	1164.683
2013	5600	24	18	0.422	0.491	3908.80	1919.221
2014	5800	12	6	0.131	0.782	4048.40	3165.849
Total							7436.353

- For 2013, the expected losses are 3908.8 = 5600(0.698) and the estimated reserves are 1919.221 = 3908.8(0.491)

- Here are the process standard deviations:

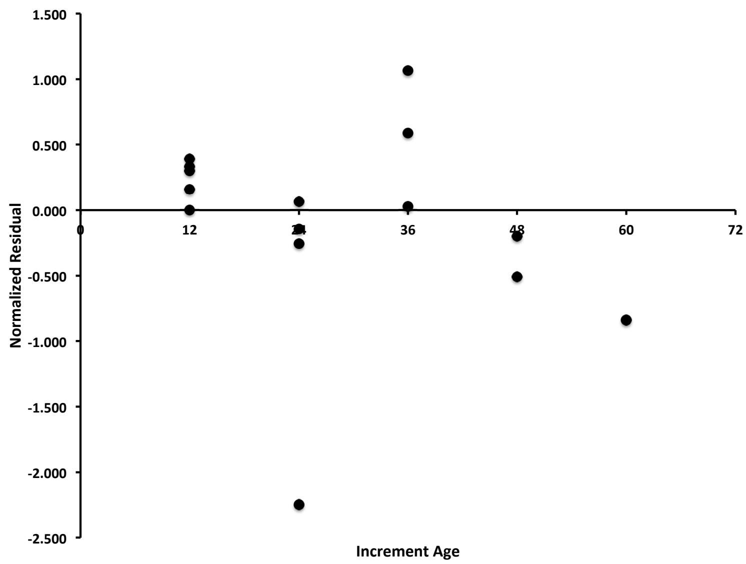
AY	Estimated Reserves	Process SD
2010	460.680	151.880
2011	725.920	190.654
2012	1164.683	241.494
2013	1919.221	$310.002 = \sqrt{50.0730(1919.221)}$
2014	3165.849	398.150
Total	7436.353	610.213

◇ Residuals

- The scale factor σ^2 is useful for a review of the model residuals, $r_{AY;x,y}$:

$$\boxed{r_{AY;x,y} = \frac{c_{AY;x,y} - \hat{\mu}_{AY;x,y}}{\sqrt{\sigma^2 \cdot \hat{\mu}_{AY;x,y}}}}$$

- We plot the residuals against a number of things to test model assumptions:
  - ◇ Increment age (i.e. AY age)
  - ◇ Expected loss in each increment - useful for testing if variance/mean ratio is constant
  - ◇ Accident year
  - ◇ Calendar year - to test diagonal effects
- In all of the cases above, we want the residuals to be randomly scattered around the zero line
- Here is an example of a residual graph for the LDF method shown above:



- In this case, the residuals do NOT appear to be randomly scattered around the zero line. Thus, we conclude that the model assumptions are invalid

◇ Testing the constant ELR assumption in the Cape Cod model

- Graph the ultimate loss ratios AY, where the ultimate loss ratio is equal to the reported losses divided by the used-up premium; this is equivalent to the loss ratios from the LDF method
- If an increasing or decreasing pattern exists, this assumption may not hold
- As an example, consider the following:

AY	On-Level Premium	Reported Losses	Growth Function	Used-Up Premium	Ultimate Loss Ratio
2014	1000	600	0.623	623	$\frac{600}{623} = 0.963$
2015	1200	500	0.472	566.4	0.883
2016	1400	180	0.178	249.2	0.722

- In this case, there is an obvious decreasing pattern in the ultimate loss ratios. Thus, the constant ELR assumption does not appear to hold

◇ Other calculations possible with this model

- Variance of the prospective losses
  - ◇ Uses the Cape Cod method
  - ◇ If we have an estimate of future year premium, we can easily calculate the estimate of expected loss (which in this case would be the estimated reserves) because we already have the maximum likelihood estimate of the ELR

- ◇ The process variance is calculated as usual
- ◇ For example, if the maximum likelihood estimate of the ELR is 0.75 and next year's planned premium is \$6M, then the prospective losses for next year are \$6M(0.75) = \$4.5M. Given $\sigma^2 = 50$, the process variance is \$4.5M(50) = \$225M

- Calendar year development
  - ◇ Rather than estimating the remaining IBNR for each accident year, we can estimate development for the next calendar year period beyond the latest diagonal
  - ◇ To estimate development for the next 12-month calendar period, we take the difference in growth functions at the two evaluation ages and multiply it by 1) the estimated ultimate losses for the loss development method OR 2) Premium*ELR for the Cape Cod method
  - ◇ The process variance and parameter variance are calculated as usual
  - ◇ A **major reason** for calculating the 12-month development is that the estimate is testable within a short timeframe. One year later, we can compare it to the actual development and see if it was within the forecast range
- Variability in discounted reserves
  - ◇ Use the same payout pattern and model parameters that were used with undiscounted reserves
  - ◇ The CV for discounted reserves is lower since the tail of the payout curve has the greatest parameter variance and also receives the deepest discount
  - ◇ See Appendix C section below for the calculation of discounted reserves, as well as an example

VI. Comments and Conclusion

◇ Abandon your triangles

- The MLE model works best when using a tabular format of data (see exhibits in paper for an example) rather than a triangular format
- All we need is a consistent aggregation of losses evaluated at more than one date

◇ The CV goes with the mean

- If we selected a carried reserve other than the maximum likelihood estimate, can we still use the CV from the model?
  - ◇ Technically, the answer is "no". The estimate of the standard deviation in the MLE model is directly tied to the maximum likelihood estimate

    - ◇ However, for practical purposes, the answer is "yes". Since the final carried reserve is a selection based on a number of factors (some of which are not captured in the model), it stands to reason that the standard deviation should also be a selection. The output from the MLE model is a reasonable basis for that selection

- ◇ Other curve forms
  - This paper focused on the loglogistic and weibull growth curves for a few reasons:
    - ◇ Smoothly move from 0% to 100%
    - ◇ Closely match the empirical data
    - ◇ First and second derivatives are calculable
  - The method is not limited to these forms; other curves could be used
- ◇ **The main conclusion of the paper is that parameter variance is generally larger than the process variance**, implying that our need for more complete data (such as the exposure information in the Cape Cod method) outweighs the need for more sophisticated models

VII. Appendix B: Adjustments for Different Exposure Periods

- ◇ Before showing the final formula, let's walk through a quick example:
  - Assume we are 9 months into an accident year
  - Then $G^*(4.5|\omega,\theta)$ represents the cumulative percent of ultimate of the 9-month period only (not the entire AY since a full AY exposure period is 12 months)
  - In order to estimate the cumulative percent of ultimate for the full accident year, we must multiply by a scaling factor that represents the portion of the AY that has been earned
  - Thus, the AY cumulative percent of ultimate as of 9 months is $G_{AY}(9|\omega,\theta) = (\frac{9}{12}) \cdot G^*(4.5|\omega,\theta)$
- ◇ Generalizing this process, there are two steps:
  - **Step 1: Calculate the percent of the period that is exposed:**

    For accident years (AY):

$$Expos(t) = \begin{cases} t/12, & t \leq 12 \\ 1, & t > 12 \end{cases}$$

  - **Step 2: Calculate the average accident date of the period that is earned:**

For accident years (AY):

$$AvgAge(t) = \begin{cases} t/2, & t \leq 12 \\ t-6, & t > 12 \end{cases}$$

◇ The final cumulative percent of ultimate curve, including annualization, is given by:

$$\boxed{G_{AY}(t|\omega,\theta) = Expos(t) \cdot G^*(AvgAge(t)|\omega,\theta)}$$

◇ *Note: Since the PY versions of the formulas above are unlikely to be tested, I have not included them*

VIII. Appendix C: Variance in Discounted Reserves

◇ Calculation of the discounted reserve, R_d:

$$\boxed{R_d = \sum_{AY}\sum_{k=1}^{y-x} ULT_{AY} \cdot v^{k-\frac{1}{2}} \cdot (G(x+k) - G(x+k-1))}$$

where $v = \frac{1}{1+i}$ and i is the constant discount rate

◇ Process variance of R_d:

$$\boxed{Var(R_d) = \sigma^2 \cdot \sum_{AY}\sum_{k=1}^{y-x} ULT_{AY} \cdot v^{2k-1} \cdot (G(x+k) - G(x+k-1))}$$

◇ **LDF method**

- For consistency, we will use the same LDF example shown earlier in the outline. Assume that expected loss emergence is described by a loglogistic curve. In addition, assume that the curve should be truncated at 120 months

- Given the following cumulative losses and parameters:

	Cumulative Losses ($)				
AY	12	24	36	48	60
2010	500	1500	2250	2590	2720
2011	550	1700	2400	2725	
2012	450	1200	2000		
2013	600	1750			
2014	575				

Parameters	
θ	21.4675
ω	1.477251
σ^2	59.9876

- We obtain the following results:

AY	Losses at 12/31/14	Age at 12/31/14	Avg. Age (x)	Growth Function	Fitted LDF	Trunc. LDF	Estimated Reserves	Estimated Ultimate
Trunc.		120	114	0.922	1.0846	1.0000		
2010	2720	60	54	0.796	1.2563	1.1583	430.576	3150.576
2011	2725	48	42	0.729	1.3717	1.2647	721.308	3446.308
2012	2000	36	30	0.621	1.6103	1.4847	969.400	2969.400
2013	1750	24	18	0.435	2.2989	2.1195	1959.125	3709.125
2014	575	12	6	0.132	7.5758	6.9848	3441.260	4016.260
Total							7521.669	17291.669

- Given a discount rate of 3%, let's determine the discounted reserves for AY 2011. To do this, we decompose AY 2011 into its CY pieces and discount them:

Age	Average Age	Growth Function	Trunc. LDF	Estimated Reserves	Discounted Reserves
Trunc.	114	0.922	1.0000	48.587	41.297
108	102	0.909	1.0143	59.892	52.433
96	90	0.893	1.0325	82.295	74.207
84	78	0.871	1.0586	115.676	107.436
72	66	0.840	1.0976	164.542	157.406
60	54	0.796	1.1583	250.315	246.643
48	42	0.729	1.2647		
				721.308	**679.421**

- Here are the calculations for age 72:
  - ◇ Avg. age $= 66 = 72 - 6$
  - ◇ Growth function $= \frac{x^\omega}{x^\omega + \theta^\omega} = \frac{66^{1.477251}}{66^{1.477251} + 21.4675^{1.477251}} = 0.840$
  - ◇ Trunc. LDF $= \frac{0.922}{0.840} = 1.0976$
  - ◇ Estimated reserves $= 3446.308\left(\frac{1}{1.0976} - \frac{1}{1.1583}\right) = 164.542$. This is the amount that emerges between ages 60 and 72

◇ Discounted reserves $= \frac{164.542}{1.03^{2-0.5}} = 157.406$. Since the average age is 66, the reserves must be discounted by 1.5 years to bring them back to the age 48 level

- *Please note that the sum of the estimated reserves over each CY piece (721.308) equals the estimated reserves found in the example shown earlier in the outline. This provides a nice check that we decomposed the reserves properly*

◇ **CC method**

- Given the following parameters for the CC method:

Parameters	
θ	22.3671
ω	1.441024
σ^2	50.0730

- As shown earlier in the outline, we obtain the following results:

AY	On-Level Premium	Age at 12/31/14	Average Age (x)	Growth Function	0.913 − Growth	Expected Losses	Estimated Reserves
Trunc.		120	114	0.913	0.000		
2010	5000	60	54	0.781	0.132	3490.00	460.680
2011	5200	48	42	0.713	0.200	3629.60	725.920
2012	5400	36	30	0.604	0.309	3769.20	1164.683
2013	5600	24	18	0.422	0.491	3908.80	1919.221
2014	5800	12	6	0.131	0.782	4048.40	3165.849
Total							7436.353

- Given a discount rate of 3%, let's determine the discounted reserves for AY 2011. To do this, we decompose AY 2011 into its CY pieces and discount them:

Age	Average Age	Growth Function	Estimated Reserves	Discounted Reserves
Trunc.	114	0.913	50.814	43.190
108	102	0.899	65.333	57.196
96	90	0.881	83.481	75.276
84	78	0.858	116.147	107.874
72	66	0.826	163.332	156.248
60	54	0.781	246.813	243.192
48	42	0.713		
			725.920	**682.976**

- Here are the calculations for age 72:
  - ◇ Avg. age = 66 = 72 − 6
  - ◇ Growth function = $\frac{x^\omega}{x^\omega+\theta^\omega} = \frac{66^{1.441024}}{66^{1.441024}+22.3671^{1.441024}} = 0.826$
  - ◇ Estimated reserves = 3629.6(0.826 − 0.781) = 163.332. This is the amount that emerges between ages 60 and 72. **Notice that we are multiplying the percentage to emerge by the expected losses, not the ultimate losses**. This is because the reserves for the CC method are based on the expected losses
  - ◇ Discounted reserves = $\frac{163.332}{1.03^{2-0.5}} = 156.248$. Since the average age is 66, the reserves must be discounted by 1.5 years to bring them back to the age 48 level
- *Please note that the sum of the estimated reserves over each CY piece (725.920) equals the estimated reserves found in the example shown earlier in the outline. This provides a nice check that we decomposed the reserves properly*

Original Mathematical Problems & Solutions

MP #1

Given the following as of December 31, 2012:

Accident Year	Reported Losses at 12/31/12	On-level Premium
2010	$7,500	$15,000
2011	6,000	15,200
2012	4,500	15,400

◇ Expected loss emergence is described by a Loglogistic curve with the following parameters:

Loglogistic Parameters	LDF Method	Cape Cod Method
ω	1.20	1.08
θ	5.50	5.45

a) Estimate the reserves as of December 31, 2012 using the LDF method with a truncation point of five years.

b) Estimate the reserves as of December 31, 2012 using the Cape Cod method with a truncation point of five years.

c) Calculate the incremental fitted payment for accident year 2012 at 12 months using the Cape Cod method.

Solution to part a:

◇ Create the following table:

AY	Losses at 12/31/12	Age at 12/31/12	Average Age (x)	Growth Function	Trunc. LDF	Estimated Reserves
Trunc. Point		60	54	0.939		
2010	7500	36	30	0.884	1.062	465
2011	6000	24	18	0.806	1.165	990
2012	4500	12	6	0.526	1.785	3532.50

- Here are the 2011 calculations for the table above:
  - ◇ Average age $= 18 = 24 - 6$
  - ◇ Growth function $= \frac{x^\omega}{x^\omega + \theta^\omega} = \frac{18^{1.2}}{18^{1.2} + 5.5^{1.2}} = 0.806$
  - ◇ Trunc. LDF $= \frac{\text{Growth function at truncation point}}{\text{Growth function at 18 months}} = \frac{0.939}{0.806} = 1.165$
  - ◇ Estimated reserves $= 6000(1.165 - 1) = 990$

◇ The total estimated reserves are $465 + 990 + 3532.50 = \boxed{\$4{,}987.50}$

Solution to part b:

◇ Calculate the expected loss ratio:

AY	On-Level Premium	Losses at 12/31/12	Age at 12/31/12	Average Age (x)	Growth Function	Premium × Growth
2010	15000	7500	36	30	0.863	12945
2011	15200	6000	24	18	0.784	11916.80
2012	15400	4500	12	6	0.526	8100.40

- Here are the 2011 calculations for the table above:
  - ◇ Average age $= 18 = 24 - 6$
  - ◇ Growth function $= \frac{x^\omega}{x^\omega + \theta^\omega} = \frac{18^{1.08}}{18^{1.08} + 5.45^{1.08}} = 0.784$
  - ◇ Premium × growth $= 15200(0.784) = 11916.80$
- The expected loss ratio is $\frac{7500 + 6000 + 4500}{12945 + 11916.80 + 8100.40} = 0.546$

◇ Estimate the reserves:

AY	On-Level Premium	Age at 12/31/14	Average Age (x)	Growth Function	0.923 − Growth	Estimated Reserves
Trunc. Point		60	54	0.923		
2010	15000	36	30	0.863	0.060	491.40
2011	15200	24	18	0.784	0.139	1153.59 = 15200(0.546)(0.139)
2012	15400	12	6	0.526	0.397	3338.13

◇ The total estimated reserves are $491.40 + 1153.59 + 3338.13 = \boxed{\$4{,}983.12}$

Solution to part c:

◇ As shown in part b. above, the ELR is 0.546

◇ The fitted incremental payment for 2012 at 12 months is ELR*Premium*(G(6)) = $0.546(15400)(0.526) = \boxed{\$4{,}422.82}$. *Note that we do not consider truncation here to calculate the fitted payment. We only use a truncated "unpaid" percentage when calculating the reserve*

MP #2

Given the following as of December 31, 2012:

Accident Year	Reported Losses at 12/31/12	On-level Premium
2010	$7,500	$15,000
2011	6,000	15,200
2012	4,500	15,400

- ◇ Expected loss emergence is described by a Weibull curve with the following parameters:

Weibull Parameters	Cape Cod Method
ω	1
θ	8

- ◇ Variance/mean ratio = 150
- ◇ Expected 2013 premium = $15,500
- ◇ The parameter covariance matrix is:

	ELR	ω	θ
ELR	0.004	-0.001	0.25
ω	-0.001	0.45	-0.30
θ	0.25	-0.30	18.00

a) Estimate the reserves as of December 31, 2012 using the Cape Cod method.

b) Calculate the process standard deviation of the 2013 expected losses using the Cape Cod method.

c) Calculate the coefficient of variation of the 2013 expected losses using the Cape Cod method.

Solution to part a:

◇ Calculate the expected loss ratio:

AY	On-Level Premium	Losses at 12/31/12	Age at 12/31/12	Average Age (x)	Growth Function	$1-$ Growth	Premium $\times$ Growth
2010	15000	7500	36	30	0.976	0.024	14640
2011	15200	6000	24	18	0.895	0.105	13604
2012	15400	4500	12	6	0.528	0.472	8131.20

- Here are the 2011 calculations for the table above:
  - ◇ Average age $= 18 = 24 - 6$
  - ◇ Growth function $= 1 - \exp(-(x/\theta)^{\omega}) = 1 - \exp(-(18/8)^1) = 0.895$
  - ◇ $1 -$ Growth $= 1 - 0.895 = 0.105$
  - ◇ Premium $\times$ growth $= 15200(0.895) = 13604$
- The expected loss ratio is $\frac{7500+6000+4500}{14640+13604+8131.20} = 0.495$

◇ Estimate the reserves:

AY	Premium $\times$ ELR	$1-$ Growth	Estimated Reserves
2010	7425	0.024	178.20
2011	$7524 = 15200(0.495)$	0.105	$790.02 = 7524(0.105)$
2012	7623	0.472	3598.06

◇ The total estimated reserves are $178.20 + 790.02 + 3598.06 = \boxed{\$4{,}566.28}$

Solution to part b:

◇ The 2013 expected losses are $15500(0.495) = 7672.50$

◇ The process variance for the 2013 expected losses is the variance/mean ratio times the expected losses

◇ Thus, the process standard deviation of the expected losses is $\sqrt{150(7672.50)} = \boxed{\$1{,}072.79}$

Solution to part c:

◇ As shown in part b., the 2013 expected losses are 7672.60 and the process variance is 150(7672.50)

◇ Parameter variance $= Var(ELR \cdot Premium) = 15500^2 \cdot Var(ELR) = 15500^2(0.004)$

- ◇ Total SD $= \sqrt{150(7672.50) + 15500^2(0.004)} = 1453.229$
- ◇ Total CoV $= 1453.229/7672.50 = \boxed{0.189}$

MP #3

Given the following as of December 31, 2012:

Accident Year	Paid Losses at 12/31/12	On-level Premium
2010	\$7,500	\$15,000
2011	6,000	15,200
2012	4,500	15,400

- ◇ Expected loss emergence is described by a Loglogistic curve with the following parameters:

Loglogistic Parameters	Cape Cod Method
ω	1.08
θ	5.45

- ◇ $i = 6\%$
- ◇ $\sigma^2 = 200$

a) Estimate the discounted reserves as of December 31, 2012 using the Cape Cod method with a truncation point of five years.

b) Calculate the process standard deviation of the 2011 discounted reserves.

Solution to part a:

◇ The discounted reserves $= \sum_{AY} \sum_{k=1}^{y-x} ULT_{AY} \cdot v^{k-\frac{1}{2}} \cdot (G(x+k) - G(x+k-1))$

◇ From part b of problem 1, we know that the 2010, 2011 and 2012 **expected** ultimate losses are 8190, 8299.20 & 8408.40, respectively (for example, 8190 = Premium x ELR = 15000(0.546))

◇ Since the truncation point is five years, $y = 60$ months = 5 years

◇ For clarity, let's consider each AY separately, starting with 2010:

Age	Average Age	Growth Function	Discounted Reserves
60	54	$0.923 = \frac{54^{1.08}}{54^{1.08}+5.45^{1.08}}$	$165.10 = \frac{8190(0.923-0.901)}{1.06^{2-0.5}}$
48	42	$0.901 = \frac{42^{1.08}}{42^{1.08}+5.45^{1.08}}$	$302.28 = \frac{8190(0.901-0.863)}{1.06^{1-0.5}}$
36	30	0.863	
			467.38

◇ Next, let's look at 2011:

Age	Average Age	Growth Function	Discounted Reserves
60	54	0.923	$157.83 = \frac{8299.20(0.923-0.901)}{1.06^{3-0.5}}$
48	42	0.901	288.98
36	30	0.863	636.81
24	18	0.784	
			1083.62

◇ Lastly, let's look at 2012:

Age	Average Age	Growth Function	Discounted Reserves
60	54	0.923	$150.86 = \frac{8408.40(0.923-0.901)}{1.06^{4-0.5}}$
48	42	0.901	276.21
36	30	0.863	608.67
24	18	0.784	2107.08
12	6	0.526	
			3142.82

◇ The total discounted reserves are $467.38 + 1083.62 + 3142.82 = \boxed{\$4,693.82}$

Solution to part b:

◇ The process variance for the discounted reserves $= \sigma^2 \cdot \sum_{AY} \sum_{k=1}^{y-x} ULT_{AY} \cdot v^{2k-1} \cdot (G(x+k) - G(x+k-1))$

◇ Let's look at 2011:

Age	Average Age	Growth Function	Process Variance Excluding σ^2
60	54	0.923	$136.44 = \frac{8299.20(0.923-0.901)}{1.06^{2(3)-1}}$
48	42	0.901	264.79
36	30	0.863	618.53
24	18	0.784	
			1019.76

◇ The process standard deviation for the reserves is $\sqrt{200(1019.76)} = \boxed{\$451.61}$

MP #4

Given the following incremental losses and reserves:

AY	Reported Losses ($) 12 mo.	24 mo.	36 mo.
2010	10,000	6,500	1,000
2011	10,500	5,500	
2012	11,000		

AY	Fitted Losses - LDF ($) 12 mo.	24 mo.	36 mo.	Reserves
2010	10,663	5,561	1,276	1,424
2011	10,516	5,484		2,663
2012	11,000			8,522

AY	Fitted Losses - Cape Cod ($) 12 mo.	24 mo.	36 mo.	Reserves
2010	10,397	5,422	1,244	1,389
2011	10,744	5,603		2,720
2012	11,090			8,592

- ◇ A loglogistic curve with two parameters was used to describe expected emergence
- ◇ Parameter variance (LDF) = $6,000,000
- ◇ Parameter variance (Cape Cod) = $3,000,000

a) Calculate the coefficient of variation of the reserves as of December 31, 2012 using the LDF method.

b) Calculate the coefficient of variation of the reserves as of December 31, 2012 using the Cape Cod method.

c) Describe how one can test the assumption that the variance/mean ratio is constant using a residual plot.

Solution to part a:

- We know that $\frac{\text{Variance}}{\text{Mean}} = \sigma^2 \approx \frac{1}{n-p}\sum_{AY,t}^{n} \frac{(c_{AY,t}-\mu_{AY,t})^2}{\mu_{AY,t}}$
- $n =$ # of data points $= 6$
- $p =$ # of parameters $= 5$ (one for each AY plus ω and θ)
- To calculate the chi-square error, we need to create the following triangle:

	Chi-Square Error:		
AY	12 mo.	24 mo.	36 mo.
2010	41.224	158.554	$59.699 = \frac{(1000-1276)^2}{1276}$
2011	0.024	0.047	
2012	0.000		

- The total chi-square error is $41.224 + 158.554 + 59.699 + 0.024 + 0.047 = 259.548$
- The variance/mean ratio is $\frac{1}{6-5}(259.548) = 259.548$
- The process variance is $\sigma^2 \cdot \text{reserves} = 259.548(1424 + 2663 + 8522) = 3{,}272{,}640.73$
- Total variance = parameter variance + process variance = $3{,}272{,}640.73 + 6{,}000{,}000 = 9{,}272{,}640.73$
- Total standard deviation = $\sqrt{9{,}272{,}640.73} = 3045.10$
- Thus, the coefficient of variation is $\frac{3045.10}{1424+2663+8522} = \boxed{0.242}$

Solution to part b:

- $n =$ # of data points $= 6$
- $p =$ # of parameters $= 3$ (ELR, ω and θ)
- To calculate the chi-square error, we need to create the following triangle:

	Chi-Square Error:		
AY	12 mo.	24 mo.	36 mo.
2010	15.159	214.328	$47.859 = \frac{(1000-1244)^2}{1244}$
2011	5.541	1.893	
2012	0.730		

- The total chi-square error is $15.159 + 214.328 + 47.859 + 5.541 + 1.893 + 0.730 = 285.510$
- The variance/mean ratio is $\frac{1}{6-3}(285.510) = 95.170$
- The process variance is $\sigma^2 \cdot \text{reserves} = 95.170(1389 + 2720 + 8592) = 1{,}208{,}754.17$

- Total variance = process variance + parameter variance = 1,208,754.17 + 3,000,000 = 4,208,754.17
- Total standard deviation = $\sqrt{4{,}208{,}754.17} = 2051.52$
- Thus, the coefficient of variation is $\frac{2051.52}{1389+2720+8592} = \boxed{0.162}$

Solution to part c:

- Plot the normalized residuals against the expected incremental losses, where the normalized residuals are equal to $\frac{\text{actual-expected}}{\sqrt{\sigma^2(\text{expected})}}$. If the normalized residuals are randomly scattered around the x-axis, then we can assume that the variance/mean ratio is constant

MP #5

Given the following as of December 31, 2012:

Accident Year	Reported Losses at 12/31/12
2010	\$13,000
2011	11,500
2012	8,000

◇ Expected loss emergence is described by a Loglogistic curve with the following parameters:

Loglogistic Parameters	LDF Method
ω	2.00
θ	4.80

a) Estimate the CY 2013 development.

b) Give a major reason for estimating next year's development.

Solution to part a:

◇ Create the following table:

AY	Losses at 12/31/12	Avg. Age at 12/31/12	Growth at 12/31/12	Avg. Age at 12/31/13	Growth at 12/31/13	Estimated Ultimate	Estimated CY 2013 Dev.
2010	13000	30	0.975	42	0.987	13333.33	160.00
2011	11500	18	0.934	30	0.975	12312.63	504.82
2012	8000	6	0.610	18	0.934	13114.75	4249.18

- Here are the 2011 calculations for the table above:
  - ◇ Growth at 12/31/12 = $\frac{18^2}{18^2+4.8^2} = 0.934$
  - ◇ Growth at 12/31/13 = $\frac{30^2}{30^2+4.8^2} = 0.975$
  - ◇ Estimated ultimate = 11500/0.934 = 12312.63
  - ◇ Estimate CY 2013 development = (0.975 − 0.934)(12312.63) = 504.82

◇ The total CY 2013 development is $160 + 504.82 + 4249.18 = \boxed{\$4{,}914}$

Solution to part b:

◇ A major reason for calculating the CY 2013 development is that the estimate is quickly testable. One year later, we can compare it to the actual development and see if it was within the forecast range

MP #6

Given the following as of September 30, 2012:

Accident Year	Reported Losses at 9/30/12
2010	$8,000
2011	6,000
2012	3,000

◇ Expected loss emergence is described by a Loglogistic curve with the following parameters:

Loglogistic Parameters	LDF Method
ω	1.40
θ	5.00

Estimate the annualized reserves as of September 30, 2012 using the LDF method.

Solution:

- ◇ Create the following table:

AY	Losses at 09/30/12	Expos(t)	Age at 09/30/12	Average Age (x)	Growth at 09/30/12	Fitted LDF	Estimated Reserves
2010	8000	1	33	27	0.914	1.094	752
2011	6000	1	21	15	0.823	1.215	1290
2012	3000	0.75	9	4.5	0.347	2.882	5646

  - Here are the 2012 calculations for the table above:
    - ◇ Expos(t) $= t/12 = 9/12 = 0.75$
    - ◇ Average age $= t/2 = 9/2 = 4.5$
    - ◇ Growth at $09/31/12 =$ Expos(t) $\cdot$ Growth function at 4.5 months $= 0.75\left(\frac{4.5^{1.4}}{4.5^{1.4}+5^{1.4}}\right) = 0.347$
    - ◇ Estimated reserves $= 3000(2.882 - 1) = 5646$

- ◇ The total estimated reserves are $752 + 1290 + 5646 = \boxed{\$7{,}688}$

Original Essay Problems

EP #1

Provide three advantages of using parameterized curves to describe loss emergence patterns.

EP #2

In a stochastic framework, explain why the Cape Cod method is preferred over the LDF method when few data points exist.

EP #3

Briefly describe the two components of the variance of the actual loss emergence.

EP #4

Provide two advantages of using the over-dispersed Poisson distribution to model the actual loss emergence.

EP #5

Fully describe the key assumptions underlying the model outlined in Clark.

EP #6

Briefly describe three graphical tests that can be used to validate Clark's model assumptions.

EP #7

Briefly explain why it might be necessary to truncate LDFs when using growth curves.

EP #8

Compare and contrast the process and parameter variances of the Cape Cod method and the LDF method.

EP #9

An actuary used maximum likelihood to parameterize a reserving model. Due to management discretion, the carried reserves differ from the maximum likelihood estimate.

a) Explain why it may NOT be appropriate to use the coefficient of variation in the model to describe the carried reserve.

b) Explain why it may be appropriate to use the coefficient of variation in the model to describe the carried reserve.

Original Essay Solutions

ES #1

- ◇ Estimation is simple since we only have to estimate two parameters
- ◇ We can use data from triangles that do NOT have evenly spaced evaluation data
- ◇ The final pattern is smooth and does not follow random movements in the historical age-to-age factors

ES #2

- ◇ The Cape Cod method is preferred since it requires the estimation of fewer parameters. Since the LDF method requires a parameter for each AY, as well as the parameters for the growth curve, it tends to be over-parameterized when few data points exist

ES #3

- ◇ Process variance – the random variation in the actual loss emergence
- ◇ Parameter variance – the uncertainty in the estimator

ES #4

- ◇ Inclusion of scaling factors allows us to match the first and second moments of any distribution. Thus, there is high flexibility
- ◇ Maximum likelihood estimation produces the LDF and Cape Cod estimates of ultimate losses. Thus, the results can be presented in a familiar format

ES #5

- ◇ Assumption 1: Incremental losses are independent and identically distributed (iid)
  - "Independence" means that one period does not affect the surrounding periods
  - "Identically distributed" assumes that the emergence pattern is the same for all accident years, which is clearly over-simplified
- ◇ Assumption 2: The variance/mean scale parameter σ^2 is fixed and known
  - Technically, σ^2 should be estimated simultaneously with the other model parameters, with the variance around its estimate included in the covariance matrix. However, doing so results in messy mathematics. For convenience and simplicity, we assume that σ^2 is fixed and known

- ◇ Assumption 3: Variance estimates are based on an approximation to the Rao-Cramer lower bound
  - The estimate of variance based on the information matrix is only exact when we are using linear functions
  - Since our model is non-linear, the variance estimate is a Rao-Cramer lower bound (i.e. the variance estimate is as low as it possibly can be)

ES #6

- ◇ Plot the normalized residuals against the following:
  - Increment age – if residuals are randomly scattered around zero with a roughly constant variance, we can assume the growth curve is appropriate
  - Expected loss in each increment age – if residuals are randomly scattered around zero with a roughly constant variance, we can assume the variance/mean ratio is constant
  - Calendar year – if residuals are randomly scattered around zero with a roughly constant variance, we can assume that there are no calendar year effects

ES #7

- ◇ For curves with heavy tails (such as loglogistic), it may be necessary to truncate the LDF at a finite point in time to reduce reliance on the extrapolation

ES #8

- ◇ Process variance – the Cape Cod method can produce a higher or lower process variance than the LDF method
- ◇ Parameter variance – the Cape Cod method produces a lower parameter variance than the LDF method since it requires fewer parameters and incorporates information from the exposure base

ES #9

Part a:

- ◇ Since the standard deviation in the MLE model is directly tied to the maximum likelihood estimate, it may not appropriate for the carried reserves

Part b:

- Since the final carried reserve is a selection based on a number of factors, it stands to reason that the standard deviation should also be a selection. The output from the MLE model is a reasonable basis for that selection

Past CAS Exam Problems & Solutions

2019 #5

A Cape Cod loss reserving calculation has the following inputs and estimates:

- ◇ Total premium is \$10,000,000
- ◇ Estimated ELR is 65%
- ◇ Process variance/mean ratio is 50,000
- ◇ The parameter covariance matrix is:

	ELR	ω	θ
ELR	0.0029	-0.0042	0.19
ω	-0.0042	0.0055	-0.41
θ	0.19	-0.41	25.52

a) Calculate the coefficient of variation of prospective losses.

b) Briefly describe what process variance and parameter variance of the prospective losses measure.

c) Briefly describe whether the Cape Cod method typically has a higher or lower parameter variance than the chain-ladder method.

Solution to part a:

- ⋄ Expected losses = 10,000,000(0.65) = 6,500,000
- ⋄ Process variance = Variance/mean ratio times the mean = 50,000(6,500,000)
- ⋄ Parameter variance = $Var(ELR{\cdot}Premium) = Premium^2{\cdot}Var(ELR) = 10,000,000^2(0.0029)$
- ⋄ Total SD = $\sqrt{50,000(6,500,000) + 10,000,000^2(0.0029)} = 784,219$
- ⋄ Total CoV = $784,219/6,500,000 = \boxed{0.121}$

Solution to part b:

- ⋄ Process variance measures uncertainty from inherent randomness of the insurance process. Parameter variance measure uncertainty in the estimated parameters

Solution to part c:

- ⋄ The Cape Cod method has a lower parameter variance because it incorporate more information from the exposure base (i.e. premium) and it uses less parameters

2019 #6

Given the following information as of December 31, 2018:

Accident Year	On-Level Earned Premium ($000,000)	Cumulative Paid Loss ($000,000) as of (months) 12 mos.	24 mos.	36 mos.
2016	13,000	360	1,425	2,850
2017	13,250	375	1,375	
2018	13,500	350		

- ◇ The expected loss payment pattern follows a loglogistic curve of the form $\frac{x^\omega}{x^\omega+\theta^\omega}$, where
  - $\omega = 1.448$
  - $\theta = 48.021$
- ◇ There are no payments after 120 months
- ◇ Accidents occur uniformly throughout the year
- ◇ The scale parameter, σ^2, is 423

a) Calculate the incremental fitted payment and corresponding normalized residual for accident year 2018 at 12 months using the Cape Cod method.

b) Calculate ultimate losses for accident year 2016 using the Cape Cod method.

Solution to part a:

- ◇ Calculate the expected loss ratio:

AY	On-Level Premium	Losses at 12/31/18	Average Age	Growth Curve	Premium × Growth
2016	13,000	2,850	30	$0.336 = \frac{30^{1.448}}{30^{1.448}+48.021^{1.448}}$	$4368 = 13000(0.336)$
2017	13,250	1,375	18	0.195	2583.75
2018	13,500	350	6	0.047	634.50

- ◇ The expected loss ratio is $\frac{350+1375+2850}{634.50+2583.75+4368} = 0.603$

- ◇ The fitted incremental payment for 2018 at 12 months is ELR*Premium*Growth $=0.603(13500)(0.047) = 382.604$

- ◇ The normalized residual is $r_{AY;x,y} = \frac{c_{AY;x,y}-\hat{\mu}_{AY;x,y}}{\sqrt{\sigma^2 \cdot \hat{\mu}_{AY;x,y}}} = \frac{350-382.604}{\sqrt{423 \cdot 382.604}} = \boxed{-0.08}$

Solution to part b:

- ◇ Truncation occurs at 120 months (avg. age of 114). The growth at 120 months is $\frac{114^{1.448}}{114^{1.448}+48.021^{1.448}} = 0.778$. Thus, the "unpaid" percentage for 2016 is $0.778 - 0.336 = 0.442$

- ◇ The 2016 reserves are $0.603(13000)(0.442) = 3464.838$

- ◇ Thus, the 2016 ultimate losses are $2850 + 3464.838 = \boxed{\$6,314,838}$

2019 #8

a) Briefly explain when a curve-fitting method for selecting loss emergence patterns will produce a higher mean estimate of ultimate losses than a weighted average method.

b) Identify one reason why each of the methods in part a. above might be better than the other for estimating the payment pattern.

c) Briefly explain why the standard deviations of the ultimate losses for each of the scenarios below are narrower than the standard deviation of the ultimate loss for the loss development method using a curve fit to derive the emerged percentages:

- ◇ Clark Cape Cod method using a curve fit to derive the emerged percentages.
- ◇ Loss development method using weighted averages of the development factors.

Solution to part a:

- ◇ Curves naturally create a tail factor by going from 0% to 100% emergence whereas weighted average methods cannot produce factors past the triangles where no data exist. This tail factor produces a higher mean estimate for the curve-fitting method

Solution to part b:

- ◇ Curve-fitting methods are better because they provide estimates of development after the end of available data
- ◇ Weighted average methods are better because they are simpler to calculate

Solution to part c:

- ◇ The Clark Cape Cod method uses an exposure base and less parameters which reduces variability of ultimate losses
- ◇ The weighted average loss development method ignores volatility in the tail which reduces variability of ultimate losses

2018 #6

Given the following information for an insurer's book of business as of December 31, 2017:

Accident Year	On-Level Premium ($000)	Cumulative Paid Loss ($000)	Estimated Reserves ($000)
2014	1,000	275	400.00
2015	1,200	306	553.85
2016	1,500	344	818.18
2017	1,700	220	1,133.33

- ◇ The estimated reserves for all accident years are calculated using the Cape Cod method
- ◇ The expected loss payment pattern is approximated by the following loglogistic function when G is the cumulative proportion of ultimate losses paid and x represents the average age of paid losses in months: $G(x) = \frac{x}{x+\theta}$

a) Calculate the expected loss ratio used in the Cape Cod method.

b) Evaluate the appropriateness of using the Cape Cod method for this book of business.

c) Briefly describe the two types of variance associated with a statistical model for loss reserving. Identify an approach to reduce one of the types of variance.

Solution to part a:

- ◇ Given the estimated reserves, we know the following:
  - AY 2014: $400 = 1000(ELR)\left(1 - \frac{42}{42+\theta}\right)$
  - AY 2015: $553.85 = 1200(ELR)\left(1 - \frac{30}{30+\theta}\right)$
  - If we divide AY 2014 by AY 2015, we have $0.722 = 0.833\left[\frac{\left(1-\frac{42}{42+\theta}\right)}{\left(1-\frac{30}{30+\theta}\right)}\right] = 0.833\left(\frac{30+\theta}{42+\theta}\right)$. Thus, $\theta = 48.05$
- ◇ Using AY 2014, we now have $400 = 1000(ELR)\left(1 - \frac{42}{42+48.05}\right)$. Thus, $ELR = \boxed{0.75}$

Solution to part b:

- ◇ Calculate the ultimate loss ratios, where the ultimate loss ratio is equal to the paid losses divided by the used-up premium:

AY	On-Level Premium	Losses at 12/31/17	Average Age	Growth Curve	Premium × Growth	Ultimate Loss Ratios
2014	1000	275	42	0.467	467	0.589
2015	1200	306	30	0.385	462	0.662
2016	1500	344	18	0.273	409.5	0.840
2017	1700	220	6	0.111	188.7	1.166

- ◇ Since the loss ratios are showing an obvious increasing pattern, there does not appear to be a constant expected loss ratio across accident years. Thus, the Cape Cod is not appropriate

Solution to part c:

- ◇ Process variance: the variance due to the randomness inherent in the insurance process
- ◇ Parameter variance: the variance due to the fact that we can't exactly estimate the parameters
- ◇ We can reduce parameter variance by the limiting the number of parameters in our model

2017 #4

Given the following data and growth curve as of December 31, 2016:

Accident Year	On-Level Premium ($000)	Reported Losses ($000)
2012	1,000	400
2013	1,300	450
2014	1,600	400
2015	1,900	250
2016	2,200	50

◇ $G(x) = \frac{x^{1.8}}{x^{1.8}+50^{1.8}}$, where G is the cumulative proportion of ultimate losses reported and x is the average age in months

Test for expected loss ratio constancy across accident years.

Solution:

◇ Calculate the ultimate loss ratios, where the ultimate loss ratio is equal to the reported losses divided by the used-up premium:

AY	On-Level Premium	Losses at 12/31/16	Average Age	Growth Curve	Premium × Growth	Ultimate Loss Ratios
2012	1000	400	54	0.535	535	0.748
2013	1300	450	42	0.422	548.6	0.820
2014	1600	400	30	0.285	456	0.877
2015	1900	250	18	0.137	260.3	0.960
2016	2200	50	6	0.022	48.4	1.033

◇ Since the loss ratios are showing an obvious increasing pattern, there does not appear to be a constant expected loss ratio across accident years

2017 #5

Given the following information as of December 31, 2016:

Accident Year	On-Level Premium	Cumulative Paid Loss
2014	\$400,000	\$210,000
2015	375,000	130,000
2016	450,000	50,000

- ◇ $G(x) = \frac{x^{1.5}}{x^{1.5}+15^{1.5}}$, where G is the cumulative proportion of ultimate losses paid and x is the average age in months
- ◇ Parameter standard deviation for Cape Cod method = 175,000
- ◇ Process variance/mean scale parameter (σ^2) for Cape Cod method = 3,000

a) Calculate the total standard deviation of the Cape Cod method's total loss reserve indication.

b) Calculate the total loss reserve by credibility-weighting the two indications from the Cape Cod method and chain-ladder method using the Benktander method.

c) Identify and briefly describe a different growth curve form that would be more appropriate to approximate the loss payment pattern for a short-tailed line of business.

Solution to part a:

◇ Calculate the expected loss ratio:

AY	On-Level Premium	Losses at 12/31/16	Average Age	Growth Curve	Premium × Growth
2014	400000	210000	30	0.739	295600
2015	375000	130000	18	0.568	213000
2016	450000	50000	6	0.202	90900

◇ The expected loss ratio is $\frac{210000+130000+50000}{295600+213000+90900} = 0.651$

◇ Estimate the reserves:

AY	Premium × ELR	1 − Growth	Estimated Reserves
2014	260400 = 400000(0.651)	0.261	67964.4 = 260400(0.261)
2015	244125	0.432	105462
2016	292950	0.798	233774.1

◇ The total estimated reserves are $67964.4 + 105462 + 233774.1 = 407200.5$

◇ The process variance is $\sigma^2 \times$ reserves. Thus, the process variance is 3000(407200.5)

◇ The total variance is process variance + parameter variance. Thus, the total variance is $3000(407200.5) + 175000^2$

◇ Thus, the total standard deviation is $\sqrt{3000(407200.5) + 175000^2} = \boxed{\$178{,}456}$

Solution to part b:

◇ Create the following table:

AY	Losses at 12/31/16	Cape Cod Reserve	Growth Curve	Chain-Ladder Reserve	Benktander Reserve
2014	210000	67964.4	0.739	74167.79	72548.71
2015	130000	105462	0.568	98873.24	101719.58
2016	50000	233774.1	0.202	197524.75	226451.73
Total					$\boxed{\$400{,}720}$

◇ Here are the calculations for AY 2014:

- Chain-ladder reserve $= \frac{210000}{0.739} - 210000 = 74167.79$
- Benktander reserve $= 74167.79(0.739) + (1 - 0.739)(67964.4) = 72548.71$

Solution to part c:

- ◇ The Weibull growth curve would be appropriate for a short-tailed line of business because it has a lighter tail (thus, it terminates sooner) than the Loglogistic curve used in the problem

2016 #3

Given the following information as of December 31, 2015:

Accident Year	On-level Premiums	Cumulative Paid Loss	Fitted Paid Emergence Pattern
2012	$500,000	$210,000	65%
2013	600,000	150,000	40%
2014	550,000	70,000	20%
2015	650,000	30,000	10%

Cape Cod Method

- ◇ Parameter standard deviation = 250,000
- ◇ Process variance/mean scale parameter (σ^2): 4,000

LDF Method

- ◇ Parameter standard deviation = 325,000
- ◇ Process variance/mean scale parameter (σ^2): 4,500

a) Calculate the total standard deviation of the total loss reserve indication resulting from the Cape Cod method.

b) Calculate the total standard deviation of the total loss reserve indication resulting from the LDF method.

c) Explain why σ^2 for the LDF method is higher than the σ^2 for the Cape Cod method.

Solution to part a:

- ◇ Calculate the expected loss ratio:

AY	On-Level Premium	Losses at 12/31/15	Growth Curve	Premium × Growth
2012	500	210	0.65	325
2013	600	150	0.40	240
2014	550	70	0.20	110
2015	650	30	0.10	65

- ◇ The expected loss ratio is $\frac{210+150+70+30}{325+240+110+65} = 0.622$
- ◇ Estimate the reserves:

AY	Premium × ELR	1 − Growth	Estimated Reserves
2012	311.00	0.35	108.85
2013	373.20 = 600(0.622)	0.60	223.92 = 373.20(0.60)
2014	342.10	0.80	273.68
2015	404.30	0.90	363.87

- ◇ The total estimated reserves are $108.85 + 223.92 + 273.68 + 363.87 = 970.32$
- ◇ The process variance is $\sigma^2 \times$ reserves. Thus, the process variance is 4000(970320)
- ◇ The total variance is process variance + parameter variance. Thus, the total variance is $4000(970320) + 250000^2$
- ◇ Thus, the total standard deviation is $\sqrt{4000(970320) + 250000^2} = \boxed{\$257{,}646}$

Solution to part b:

- ◇ Create the following table:

AY	Losses at 12/31/15	Growth Curve	Reserves
2012	210	0.65	$113.08 = \frac{210}{0.65} - 210$
2013	150	0.40	225
2014	70	0.20	280
2015	30	0.10	270

- ◇ The total estimated reserves are $113.08 + 225 + 280 + 270 = 888.08$
- ◇ The process variance is $\sigma^2 \times$ reserves. Thus, the process variance is 4500(888080)

- ◇ The total variance is process variance + parameter variance. Thus, the total variance is $4500(888080) + 325000^2$
- ◇ Thus, the total standard deviation is $\sqrt{4500(888080) + 325000^2} = \boxed{\$331{,}091}$

Solution to part c:

- ◇ The σ^2 refers to the process variance. When calculating σ^2, we divide by $n - p$, where p is the number of parameters. Since the LDF method requires more parameters, it has a higher σ^2.

2016 #4

Given the following information for an insurer's book of business as of December 31, 2015:

Accident Year	On-Level Premium ($000)	Paid Losses ($000)
2012	800	480
2013	1,000	530
2014	1,500	640
2015	1,250	290

- The expected loss payment pattern for the insurance company was approximated by the following function, where G is the cumulative proportion of ultimate losses paid and x represents the average age (in months) since accident occurrence:

$$G(x) = \frac{x^{1.1}}{x^{1.1} + 8.0^{1.1}}$$

- The expected loss ratio (ELR) is 62.5% for this book

a) Use the Cape Cod method to calculate the expected unpaid losses for accident year 2013.

b) Evaluate the appropriateness of using the Cape Cod method with a constant ELR for this book of business.

Solution to part a:

◇ Create the following table:

AY	Age	Avg. Age	ELR	Premium × ELR	Growth	Reserve
2013	36	30	0.625	$625 = 1000(0.625)$	$0.811 = \frac{30^{1.1}}{30^{1.1}+8^{1.1}}$	$118.125 = 625(1-0.811)$

◇ The AY 2013 reserve is $\boxed{\$118{,}125}$

Solution to part b:

◇ To evaluate the appropriateness of using the Cape Cod method with a constant ELR, we should calculate the ultimate loss ratios, where the ultimate loss ratio is equal to the reported losses divided by the used-up premium:

AY	Premium	Age	Avg. Age	Growth	Premium × Growth	Paid Loss	Ultimate Loss Ratios
2012	800	8	42	$0.861 = \frac{42^{1.1}}{42^{1.1}+8^{1.1}}$	$688.8 = 800(0.861)$	480	$0.697 = \frac{480}{688.8}$
2013	1000	36	30	0.811	811	530	0.654
2014	1500	24	18	0.709	1063.5	640	0.602
2015	1250	12	6	0.422	527.5	290	0.550

◇ Since the loss ratios show an obvious downward trend, a constant ELR will overstate reserves for recent years and understate reserves for older years. Thus, a constant ELR is **NOT appropriate**

2015 #2

Given the following paid claim information as of December 31, 2014:

Accident Year	Paid Claims ($000)
2011	12,000
2012	11,250
2013	14,750
2014	9,500
Total	47,500

- ◇ The expected paid claim emergence pattern has been approximated by the following function where G is the cumulative proportion of ultimate claims paid and x represents the average time since accident occurrence in months.

$$G(x) = \frac{x}{x+10}$$

- ◇ The expected incremental paid claim emergence follows an over-dispersed Poisson distribution with scaling factor $\sigma^2 = 25000$
- ◇ Parameter standard deviation for the total estimated unpaid claims is $850,000

a) Using a truncation point of 10 years, calculate the coefficient of variation of the total unpaid claims using the LDF method.

b) Identify the direction in which the coefficient of variation of the total unpaid claims estimate would change if the method used to calculate the unpaid claims estimate were changed from the LDF method to the Cape Cod method, and briefly explain the reason it would change in this direction.

Solution to part a:

◇ Create the following table:

AY	Losses at 12/31/14	Age at 12/31/14	Average Age (x)	Growth Function	Trunc. LDF	Estimated Reserves
Trunc. Point		120	114	0.919		
2011	12000	48	42	0.808	1.137	1644.00
2012	11250	36	30	0.750	1.225	2531.25
2013	14750	24	18	0.643	1.429	6327.75
2014	9500	12	6	0.375	2.451	13784.50

- Here are the 2013 calculations for the table above:
  - ◇ Average age $= 18 = 24 - 6$
  - ◇ Growth function $= \frac{x}{x+10} = \frac{18}{18+10} = 0.643$
  - ◇ Trunc. LDF $= \frac{\text{Growth function at truncation point}}{\text{Growth function at 18 months}} = \frac{0.919}{0.643} = 1.429$
  - ◇ Estimated reserves $= 14750(1.429 - 1) = 6427.75$
- The total estimated reserves are $1644 + 2531.25 + 6327.75 + 13784.50 = 24287.50$
- The total process variance is $24287.50(\sigma^2) = 24287.50(25)$
- The total parameter variance is 850^2
- The total standard deviation is $\sqrt{24287.50(25) + 850^2} = 1153.121$
- Thus, the total coefficient of variation is $\frac{1153.121}{24287.50} = \boxed{0.0475}$

Solution to part b:

◇ The CV will be reduced. This is because we are relying on more information like premium or exposure, and this information allows us to make significantly better estimate of the reserve

2014 #3

Given the following data for a Cape Cod reserve analysis:

Accident Year	Actual Incremental Reported Losses ($000) 12 Months	24 Months	36 Months
2010	100	255	180
2011	120	280	
2012	120		

Accident Year	Expected Incremental Reported Losses ($000) 12 Months	24 Months	36 Months
2010	80	300	200
2011	80	320	
2012	100		

The parameters of the loglogistic growth curve (ω and θ) and the expected loss ratio (ELR) were previously estimated, resulting in a total estimated reserve of $1,500,000. The parameter standard deviation of the total estimated reserve is $350,000.

Calculate the standard deviation of the reserve due to parameter and process variance combined.

Solution:

- ◇ We know that $\frac{\text{Variance}}{\text{Mean}} = \sigma^2 \approx \frac{1}{n-p} \sum_{AY,t}^{n} \frac{(c_{AY,t} - \mu_{AY,t})^2}{\mu_{AY,t}}$
- ◇ $n =$ # of data points $= 6$
- ◇ $p =$ # of parameters $= 3$ (ELR, ω and θ)
- ◇ To calculate the chi-square error, we need to create the following triangle:

	Chi-Square Error:		
AY	12 mo.	24 mo.	36 mo.
2010	5	6.75	$2 = \frac{(180-200)^2}{200}$
2011	20	5	
2012	4		

- ◇ The total chi-square error is $5 + 6.75 + 2 + 20 + 5 + 4 = 42.75$
- ◇ The variance/mean ratio is $\frac{1}{6-3}(42.75) = 14.25$. Since the numbers in the table above are in thousands, we convert this to 14250
- ◇ The process variance is $\sigma^2 \cdot \text{reserves} = 14250(1500000)$
- ◇ Total variance = parameter variance + process variance $= 350000^2 + 14250(1500000)$
- ◇ Total standard deviation $= \sqrt{350000^2 + 14250(1500000)} = \boxed{\$379{,}308.58}$

2014 #5

An insurance company has 1,000 exposures uniformly distributed throughout the accident year. The a priori ultimate loss is $800 per exposure unit.

The expected loss payment pattern is approximated by the following loglogistic function where G is the cumulative proportion of ultimate losses paid and x represents the average age of reported losses in months.

- ⋄ $G(x) = \frac{x^\omega}{x^\omega + \theta^\omega}$
- ⋄ $\omega = 2.5$
- ⋄ $\theta = 24$

a) Calculate the expected losses paid in the first 36 months after the beginning of the accident year.

b) Assume the actual cumulative paid losses at 36 months after the beginning of the accident year are $650,000. Estimate the ultimate loss for the accident year using assumptions based upon the Cape Cod method.

c) Estimate the ultimate loss for the accident year based on the loglogistic payment model and the actual payments through 36 months, disregarding the a priori expectation.

d) Calculate a reserve estimate for the accident year by credibility-weighting two estimates of ultimate loss in parts b. and c. above using the Benktander method.

Solution to part a:

- ◇ At 36 months after the beginning of the accident year, the average age of the reported losses is 30 months
- ◇ $G(30) = \frac{30^{2.5}}{30^{2.5}+24^{2.5}} = 0.636$
- ◇ Expected losses $= 1000(800)(0.636) = \boxed{\$508{,}800}$

Solution to part b:

- ◇ Ultimate loss $=$ paid $+$ IBNR $= 650000 + 1000(800)(1 - 0.636) = \boxed{\$941{,}200}$
- ◇ *Note: I am not a fan of the wording in this part. The problem says "based upon the Cape Cod method", but this is more of a BF problem where we use the a priori loss to inform the IBNR. As an exam taker, use the other parts to help you understand what the CAS is asking for. In part d., they ask for a Benktander credibility weighting between parts b. and c. With this in mind, we can deduce that part b. must be asking for a BF ultimate loss*

Solution to part c:

- ◇ $\frac{650000}{0.636} = \boxed{\$1{,}022{,}013}$

Solution to part d:

- ◇ For the Benktander method, $Z = p_k = G(30) = 0.636$
- ◇ Ultimate loss $= 1022013(0.636) + (1 - 0.636)(941200) = 992597$
- ◇ Reserve $= 992597 - 650000 = \boxed{\$342{,}597}$

2013 #3

Given the following information:

	Cumulative Paid Loss ($000)		
Accident Year	12	24	36
2010	2,750	4,250	5,100
2011	2,700	4,300	
2012	2,900		

- ◇ The expected accident year loss emergence pattern (growth function) is approximated by a Weibull function of the form:

$$G(x|\omega, \theta) = 1 - exp(-(x/\theta)^{\omega})$$

- ◇ Parameter estimates are: $\omega = 1.5$ and $\theta = 20$

a) Calculate the process standard deviation of the reserve estimate for accident years 2010 through 2012 using the LDF method.

b) Calculate the normalized residuals for all six data cells in the triangle above. (*Note: I modified this part since the original problem asked you to create a graph. You should know how to interpret residual plots from Clark.*

Solution to part a:

◇ Calculate the reserves

- Create the following table:

AY	Losses at 12/31/12	Age at 12/31/12	Average Age (x)	Growth Function	LDF	Estimated Reserves
2010	5100	36	30	0.841	1.189	963.90
2011	4300	24	18	0.574	1.742	3190.60
2012	2900	12	6	0.152	6.579	16179.10

- Here are the 2011 calculations for the table above:
  - ◇ Average age $= 18 = 24 - 6$
  - ◇ Growth function $= 1 - \exp(-(x/\theta)^{\omega}) = 1 - \exp(-(18/20)^{1.5}) = 0.574$
  - ◇ LDF $= \frac{1}{0.574} = 1.742$
  - ◇ Estimated reserves $= 4300(1.742 - 1) = 3190.60$
- The total estimated reserves are $963.90 + 3190.60 + 16179.10 = 20333.60$

◇ Calculate the process standard deviation

- Create the fitted incremental triangle:

	Fitted Incremental Losses:		
AY	12 mo.	24 mo.	36 mo.
2010	$921.713 = 0.152(5100 + 963.9)$	2558.966	1619.061
2011	1138.571	3161.033	
2012	2900.023		

- Create the chi-square error incremental triangle:

	Chi-Square Error:		
AY	12 mo.	24 mo.	36 mo.
2010	$3626.545 = \frac{(2750-921.713)^2}{921.713}$	438.227	365.307
2011	2141.334	770.895	
2012	0.000		

- The total chi-square error is $3626.545 + 438.227 + 365.307 + 2141.334 + 770.895 = 7342.308$
- We know that $\frac{\text{Variance}}{\text{Mean}} = \sigma^2 \approx \frac{1}{n-p} \sum_{AY,t}^{n} \frac{(c_{AY,t}-\mu_{AY,t})^2}{\mu_{AY,t}}$

- $n =$ # of data points $= 6$
- $p =$ # of parameters $= 5$ (one for each AY plus ω and θ)
- The variance/mean ratio is $\frac{1}{6-5}(7342.308) = 7342.308$
- The process standard deviation is $\sqrt{\sigma^2 \cdot \text{reserves}} = \sqrt{7342.308(20333.60)} = \boxed{\$12{,}218{,}656}$

Solution to part b:

◇ The normalized residual, $r_{AY;x,y} = \frac{c_{AY;x,y} - \hat{\mu}_{AY;x,y}}{\sqrt{\sigma^2 \cdot \hat{\mu}_{AY;x,y}}}$. Using this formula, we can create the following normalized residual triangle:

Normalized Residuals:

AY	12 mo.	24 mo.	36 mo.
2010	0.703	-0.244	-0.223 $= \frac{(850-1619.061)}{\sqrt{7342.308(1619.061)}}$
2011	0.540	-0.324	
2012	0.000		

2012 #2

Given the following information as of December 31, 2011:

Accident Year	On-level Premiums	Cumulative Paid Loss	Fitted Paid Emergence Pattern
2008	\$1,300,000	\$600,000	70%
2009	1,200,000	350,000	45%
2010	1,200,000	200,000	25%
2011	1,300,000	75,000	10%

◇ Parameter standard deviation: 300,000

◇ Process variance/scale parameter (σ^2): 10,000

a) Estimate the total loss reserve using the Cape Cod method.

b) Calculate the process standard deviation of the reserve estimate in part a. above.

c) Calculate the total standard deviation and the coefficient of variation of the reserve estimate.

Solution to part a:

- ⋄ Calculate the expected loss ratio:

AY	On-Level Premium	Losses at 12/31/12	Growth Function	Premium × Growth
2008	1300	600	0.70	910
2009	1200	350	0.45	540
2010	1200	200	0.25	300
2011	1300	75	0.10	130

- ⋄ The expected loss ratio is $\frac{600+350+200+75}{910+540+300+130} = 0.652$
- ⋄ Estimate the reserves:

AY	Premium × ELR	1 − Growth	Estimated Reserves
2008	847.60	0.30	254.28
2009	782.40 = 1200(0.652)	0.55	430.32 = 782.40(0.55)
2010	782.40	0.75	586.80
2011	847.60	0.90	762.84

- ⋄ The total estimated reserves are $254.28 + 430.32 + 586.80 + 762.84 = \boxed{\$2{,}034{,}240}$

Solution to part b:

- ⋄ Process variance $= \sigma^2 \times$ reserves $= 10000(2{,}034{,}240)$
- ⋄ Process standard deviation $= \sqrt{10000(2{,}034{,}240)} = \boxed{\$142{,}626.79}$

Solution to part c:

- ⋄ Total variance = process variance + parameter variance $= 10000(2{,}034{,}240) + 300000^2$
- ⋄ Total standard deviation $= \sqrt{10000(2{,}034{,}240) + 300000^2} = 332178.265$
- ⋄ Thus, the coefficient of variation $= \frac{332178.265}{2{,}034{,}240} = \boxed{0.163}$

2011 #2

Given the following loss reserving information as of December 31, 2010:

Accident Year	On-Level Earned Premium	Growth Function	Reported Losses
2008	\$13,500	78.9%	\$7,200
2009	14,000	57.9%	5,700
2010	14,500	13.8%	1,400
Total	42,000		14,300

- Parameter standard deviation for the total estimated unpaid claims is 796
- The expected accident year loss emergence pattern (growth function) can be approximated by a loglogistic function of the form:

$$G(x|\omega, \theta) = x^{\omega}/(x^{\omega} + \theta^{\omega}),$$

  where x denotes time in months from the average accident date to the evaluation date, and G is the growth function describing cumulative percent reported
- The maximum likelihood estimates of the parameters are:

$$\omega = 1.956 \text{ and } \theta = 15.286$$

- The actual incremental loss emergence follows an over-dispersed Poisson distribution with scaling factor $\sigma^2 = 9$

a) Using a truncation point of five years, estimate the total unpaid claims using the Cape Cod method.

b) Calculate the coefficient of variation of the total unpaid claims estimated in part a. above.

Solution to part a:

◇ Calculate the expected loss ratio:

AY	On-Level Premium	Losses at 12/31/10	Age at 12/31/10	Average Age (x)	Growth Function	Premium × Growth
2008	13500	7200	36	30	0.789	10651.50
2009	14000	5700	24	18	0.579	8106.00
2010	14500	1400	12	6	0.138	2001.00

- Here are the 2009 calculations for the table above:
  - ◇ Average age = 18 = 24 − 6
  - ◇ Growth function $= \frac{x^\omega}{x^\omega + \theta^\omega} = \frac{18^{1.956}}{18^{1.956} + 15.286^{1.956}} = 0.579$
  - ◇ Premium × growth = 14000(0.579) = 8106
- The expected loss ratio is $\frac{7200+5700+1400}{10651.50+8106+2001} = 0.689$

◇ Estimate the reserves:

AY	On-Level Premium	Age at 12/31/14	Average Age (x)	Growth Function	0.922 − Growth	Estimated Reserves
Trunc. Point		60	54	0.922		
2008	13500	36	30	0.789	0.133	1237.100
2009	14000	24	18	0.579	0.343	3308.578 = 14000(0.689)(0.343)
2010	14500	12	6	0.138	0.784	7832.552

◇ The total estimated reserves are $1237.100 + 3308.578 + 7832.552 = \boxed{\$12{,}378.23}$

Solution to part b:

◇ Process variance $= \sigma^2 \times$ reserves $= 9(12378.23) = 111404.07$

◇ Total variance = process variance + parameter variance $= 111404.07 + 796^2 = 745020.07$

◇ Total standard deviation $= \sqrt{745020.07} = 863.145$

◇ Thus, the coefficient of variation $= \frac{863.145}{12378.23} = \boxed{0.0697}$

