Rising Fellow

2024

Exam 7

Study Guide

ADVANCED ESTIMATION OF CLAIMS LIABILITIES

Comprehensive study guide
with original and past CAS problems

Exam 7 Study Guide

2024 Sitting

Rising Fellow

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Introduction

How To Use This Guide

This guide is intended to **supplement** the syllabus readings. Although we believe it provides a thorough review of the exam material, the readings provide additional context that is invaluable. Please do NOT skip the syllabus readings.

Original Mathematical & Essay Problems

Original mathematical & essay problems/solutions are included for all papers. If a topic is covered in an essay problem, then you should know it. All original practice problems are included in the guide and as separate Excel workbooks. The Excel workbooks can be downloaded from the online course.

Past CAS Exam Problems

Past CAS exam problems & solutions are included for each paper. Note that these questions are solely owned by the CAS. They are included in the online course for student convenience. All past CAS problems are included in the guide and as separate Excel workbooks. The Excel workbooks can be downloaded from the online course.

Feedback

We always working to improve the Exam 7 Study Guide and the rest of the Rising Fellow study material. Please send us an email at exam7@risingfellow.com if you have feedback about any of the following:

- ◇ Sections that are confusing or could be improved
- ◇ Errors (ex. formatting, spelling, calculations, grammar, etc.)

Note that errata will be posted on the Rising Fellow website on an as-needed basis.

Blank Pages

Since many students want a printed copy of the study guide, blank pages have been inserted throughout the guide to ensure that all outlines start on odd pages.

Bookmarks

Bookmarks have been added for each section listed in the table of contents for easier navigation in Adobe Acrobat.

Mack (2000)

Outline

◇ Notation

- p_k is the proportion of the ultimate claims amount which is expected to be paid after k years of development
- $q_k = 1 - p_k$ is the proportion of the ultimate claims amount which is expected to remain unpaid after k years of development
- $U_0 = U^{(0)}$ is the a priori expectation of ultimate losses (i.e. expected ultimate losses)
- $U_{BF} = U^{(1)}$ is the Bornhuetter/Ferguson ultimate claims estimate
- $U_{GB} = U^{(2)}$ is the Gunner Benktander ultimate claims estimate
- $U_{CL} = U^{(\infty)}$ is the chain ladder ultimate claims estimate
- $U^{(m)}$ is the ultimate claim estimate at the m^{th} iteration
- U_c is a credibility weighted ultimate claims estimate, where c is the credibility factor
- $\hat{U}$ is any ultimate claims estimate
- R_{BF} is the Bornhuetter/Ferguson reserve estimate
- R_{CL} is the chain ladder reserve estimate
- R_{GB} is the Gunner Benktander reserve estimate
- $\hat{R}$ is any reserve estimate
- C_k is the actual claims amount paid after k years of development

◇ General relationship between any reserve estimate $\hat{R}$ and the corresponding ultimate claims estimate $\hat{U}$:

$$\boxed{\hat{U} = C_k + \hat{R}}$$

◇ **Bornhuetter/Ferguson method**

- Reserve estimate based on the a priori expectation of ultimates losses:

$$\boxed{R_{BF} = q_k U_0}$$

- Using the general relationship described earlier, $U_{BF} = C_k + R_{BF}$

- Since R_{BF} uses U_0, it assumes the current claims amount C_k is not predictive of future claims

◇ **Chain ladder method**

- $U_{CL} = C_k/p_k$
- Using the general relationship described earlier, $R_{CL} = U_{CL} - C_k$
- Combining the two previous formulae, it can be shown that

$$\boxed{R_{CL} = q_k U_{CL}}$$

- Since R_{CL} uses U_{CL}, it assumes the current claims amount C_k is fully predictive of future claims
- **Advantage of *CL* over *BF***: Using *CL*, different actuaries obtain similar results. This is not the case with *BF* due to differences in the selection of U_0

◇ **Benktander method**

- Also known as Iterated Bornhuetter/Ferguson method
- Since *CL* and *BF* represent extreme positions (fully believe C_k vs. do not believe at all), Benktander replaced U_0 with a credibility mixture:

$$\boxed{U_c = cU_{CL} + (1-c)U_0}$$

- As the claims C_k develop, credibility should increase. As a result, Benktander proposed setting $c = p_k$ and estimating the claims reserve by $R_{GB} = R_{BF} \cdot \frac{U_{p_k}}{U_0}$
- Combining this with the formula for R_{BF}, we can easily show that $R_{GB} = q_k U_{p_k}$
- Using our credibility mixture, we can show that $U_{p_k} = p_k U_{CL} + q_k U_0 = C_k + R_{BF} = U_{BF}$, which finally brings us to the following:

$$\boxed{R_{GB} = q_k U_{BF}}$$

- This equation has the following implications:
  - ◇ R_{GB} is obtained by applying the *BF* procedure twice, first to U_0, and then to U_{BF} (hence, the Iterated Bornhuetter/Ferguson method)
  - ◇ The Benktander method is a credibility weighted average of the *BF* method and the *CL* method, where $c = p_k = 1 - q_k$:

$$\boxed{\begin{aligned} U_{GB} &= C_k + R_{GB} \\ &= (1-q_k)U_{CL} + q_k U_{BF} \end{aligned}}$$

- Note: $U_{GB} = C_k + R_{GB} = (1 - q_k^2)U_{CL} + q_k^2 U_0 = U_{1-q_k^2} \neq U_{p_k}$, which illustrates the fact that the BF method and GB produce different results. It also shows that the Benktander method is a credibility weighted average of the CL method and the a priori expectation of ultimate losses, where $c = 1 - q_k^2$

- It is also possible to apply the credibility mixture directly to the reserves instead of the ultimates. Esa Hovinen proposed the following reserve estimate: $R_{EH} = cR_{CL} + (1 - c)R_{BF}$. If we set $c = p_k$ as before, we find that $R_{EH} = R_{GB}$

◇ In his paper, Mack presents a theorem that shows how ultimates and reserves change as we iterate through indefinitely (rather than just iterating twice for the GB method). Since I don't think it's worth memorizing for the exam, let's just get to the results. Using the iteration rules $R^{(m)} = q_k U^{(m)}$ and $U^{(m+1)} = C_k + q_k U^{(m)}$, we obtain the following credibility mixtures:

$$\boxed{\begin{aligned} U^{(m)} &= (1 - q_k^m)U_{CL} + q_k^m U_0 \\ R^{(m)} &= (1 - q_k^m)R_{CL} + q_k^m R_{BF} \end{aligned}}$$

◇ If we iterate between reserves and ultimates indefinitely, we will eventually end up with the CL result

◇ The Benktander method is superior to BF and CL for a few reasons:

- **Lower mean squared error (MSE)**

  ◇ Walter Neuhaus compared the MSE of $R_c = cR_{CL} + (1 - c)R_{BF}$ for $c = 0$ (BF), $c = p_k$ (GB), and $c = c^*$ (optimal credibility reserve that minimizes the MSE)

  ◇ MSE of R_{GB} is smaller than MSE of R_{BF} when $c^* > p_k/2$. This makes sense because the inequality implies that c^* is closer to $c = p_k$ than to $c = 0$

  ◇ Mack also states in the abstract that the Benktander method almost always has a smaller MSE than BF and CL

- **Better approximation of the exact Bayesian procedure**

- **Superior to *CL* since it gives more weight to the a priori expectation of ultimate losses**

- **Superior to *BF* since it gives more weight to actual loss experience**

Original Mathematical Problems & Solutions

MP #1

Given the following information for accident year 2012 as of December 31, 2012:

- ◇ 12-ultimate cumulative paid LDF = 1.60
- ◇ Ultimate loss based on the chain-ladder method = $12,000
- ◇ Ultimate loss based on the Benktander method = $14,000

Calculate the accident year 2012 ultimate loss based on the Bornhuetter/Ferguson method.

Solution:

- $U_{GB} = (1 - q_k)U_{CL} + q_k U_{BF}$
- $q_k = 1 - p_k = 1 - \frac{1}{\text{LDF}} = 1 - \frac{1}{1.6} = 0.375$
- Plugging q_k into our formula for U_{GB}, we have 14000 = (1 - 0.375)12000 + 0.375(U_{BF})
- Thus, $\boxed{U_{BF} = \$17{,}333.33}$

MP #2

Given the following:

	Cumulative Paid Losses ($)			
AY	12 mo.	24 mo.	36 mo.	48 mo.
2009	7,000	10,500	12,600	13,860
2010	8,000	12,000	14,400	
2011	9,000	13,500		
2012	10,000			

- ◇ The 2010 earned premium is $25,000
- ◇ The expected loss ratio for each year is 75%
- ◇ Assume the 48-ultimate loss development factor is 1.05

Calculate the accident year 2010 ultimate loss based on the Benktander method.

Solution:

- ◇ $U_{GB} = C_k + R_{GB}$
- ◇ From the loss triangle, $C_k = 14400$
- ◇ We need to calculate $R_{GB} = q_k U_{BF}$
- ◇ To determine q_k, we need to calculate the 36-ultimate LDF:
  - The 36-48 LDF is $13860/12600 = 1.10$
  - Combining this with the 48-ultimate LDF gives a 36-ultimate LDF of $(1.10)(1.05) = 1.155$
  - Then, $q_k = 1 - \frac{1}{1.155} = 0.134$
- ◇ To determine U_{BF}, we need to calculate U_0 for 2010:
  - $U_0 = EP \cdot ELR = 25000(0.75) = 18750$
  - $U_{BF} = C_k + R_{BF} = C_k + q_k U_0 = 14400 + 0.134(18750) = 16912.50$
- ◇ We can now calculate $R_{GB} = 0.134(16912.50) = 2266.275$
- ◇ Finally, $U_{GB} = 14400 + 2266.275 = \boxed{\$16,666.28}$

MP #3

Given the following information for accident year 2012 as of December 31, 2012:

- ◇ $U_0 = \$5{,}000$
- ◇ $C_k = \$3{,}000$
- ◇ $q_k = 0.60$

a) Calculate $U^{(3)}$.

b) Calculate $U^{(\infty)}$.

Solution to part a:

- ◇ $U^{(1)} = U_{BF} = C_k + q_k U_0 = 3000 + 0.6(5000) = 6000$
- ◇ $U^{(2)} = U_{GB} = C_k + q_k U_{BF} = 3000 + 0.6(6000) = 6600$
- ◇ $U^{(3)} = C_k + q_k U_{GB} = 3000 + 0.6(6600) = \boxed{\$6{,}960}$

Solution to part b:

- ◇ $U^{(\infty)} = U_{CL} = C_k / p_k = 3000/(1 - 0.6) = \boxed{\$7{,}500}$

MP #4

Given the following information for accident year 2012 as of December 31, 2012:

- 12-ultimate cumulative paid LDF = 2.50
- Reserve based on the chain-ladder method = $4,000
- Ultimate loss based on the Benktander method = $8,000

Using a credibility weight of $c = p_k$, calculate the accident year 2012 Esa Hovinen reserve.

Solution:

- When $c = p_k$, $R_{EH} = R_{GB} = U_{GB} - C_k$
- To determine C_k:
  - $R_{CL} = q_k U_{CL}$
  - $U_{CL} = 4000/(1 - \frac{1}{2.5}) = 6666.667$
  - Thus, $C_k = U_{CL} - R_{CL} = 6666.667 - 4000 = 2666.667$
- Plugging C_k into our formula for R_{EH}, we find that $R_{EH} = 8000 - 2666.667 = \boxed{\$5{,}333.33}$

MP #5

Given the following information for accident year 2012 as of December 31, 2012:

- $c^* = 0.32$
- $C_k = \$3,000$
- $U_{CL} = \$5,000$

Which reserve has a smaller MSE: R_{GB} or R_{BF}?

Solution:

- ◇ $U_{CL} = C_k/p_k$. Thus, $p_k = 0.6$
- ◇ If $c^* > p_k/2$, R_{GB} has a smaller MSE
- ◇ Checking the condition above, $0.32 > 0.6/2$
- ◇ Thus, $\boxed{R_{GB} \text{ has a smaller MSE}}$

Past CAS Exam Problems & Solutions

2018 #5

Given the following information about accident year 2017 as of December 31, 2017:

- ◇ Accident year 2017 paid loss = $850,000
- ◇ 2017 earned premium = $4,000,000
- ◇ Initial expected loss ratio = 67.5%
- ◇ 12-24 month incremental paid link ratio = 1.60
- ◇ 12-ultimate cumulative paid LDF = 3.00

a) Determine the accident year 2017 incremental paid loss in 2018 that would result in the Benktander ultimate loss estimate being $100,000 less than the Bornhuetter-Ferguson ultimate loss estimate for accident year 2017 as of December 31, 2018. Assume all development factors are unchanged.

b) Briefly describe when the Benktander ultimate loss estimate would be greater than the Bornhuetter-Ferguson ultimate loss estimate as of December 31, 2018.

c) Explain why it may not be appropriate to use the Bornhuetter-Ferguson method when losses develop downward.

Solution to part a:

- ◇ $U_{BF} = C_K + U_0 q_k = (850 + x) + 4000(0.675)\left(1 - \frac{1}{3/1.6}\right) = 2110 + x$. *Notice here that we are dividing 3 by 1.6 to obtain the cumulative paid LDF at 24 months*
- ◇ $U_{GB} = C_k + U_{BF} q_k = (850 + x) + (2110 + x)\left(1 - \frac{1}{3/1.6}\right)$. Since we want U_{GB} to be 100,000 less than U_{BF}, we have $(850 + x) + (2110 + x)\left(1 - \frac{1}{3/1.6}\right) = 2110 + x - 100$. Thus, $x = \boxed{\$375{,}714}$

Solution to part b:

- ◇ Since the Benktander estimate is a weighting of the CL estimate and the BF estimate, the Benktander estimate is greater than the BF estimate when the CL estimate is greater than the BF estimate

Solution to part c:

- ◇ Since the BF IBNR does not respond to actual loss performance, the downward development will not affect IBNR produced by the BF method. If the downward development represents real trends (such as increased salvage and subrogation), then the BF method will overstate the IBNR

2013 #4

Given the following information:

AY	Cumulative Paid Loss ($000) 12 mo.	24 mo.	36 mo.	48 mo.
2009	5,751	10,640	11,491	12,181
2010	5,528	9,287	10,680	
2011	4,120	7,004		
2012	5,304			

Accident Year	Calculated Ultimate Loss ($000) Bornhuetter/Ferguson Ultimate	Benktander Ultimate
2009	12,181	12,181
2010	11,246	11,316
2011	8,428	8,204
2012	10,403	10,609

a) Calculate the 24-month-to-ultimate cumulative development factor that would result in the ultimate loss estimates shown above.

b) For accident year 2011, suppose that the Bornhuetter/Ferguson method is performed over multiple iterations. Deduce the ultimate loss estimate that will be produced as the number of iterations approaches infinity.

Solution to part a:

- ⋄ Since we want to calculate the 24-ultimate development factor, let's look at AY 2011
- ⋄ $U_{GB} = C_k + q_k U_{BF}$
- ⋄ $8204 = 7004 + q_k(8428)$
- ⋄ $q_k = 0.142$
- ⋄ $0.142 = 1 - \frac{1}{LDF_{24-ult}}$
- ⋄ Thus, $LDF_{24-ult} = \boxed{1.166}$

Solution to part b:

- ⋄ As the number of Bornhuetter/Ferguson iterations approaches infinity, the chain-ladder ultimate loss estimate will be produced

2012 #1

Given the following information for accident year 2011 as of December 31, 2011:

- ◇ Accident year 2011 paid loss = \$700,000
- ◇ 2011 earned premium = \$3,000,000
- ◇ Initial expected loss ratio = 62.5%
- ◇ 12-24 month paid link ratio = 1.50
- ◇ 12-ultimate cumulative paid LDF = 2.50

a) Calculate accident year 2011 ultimate loss estimates as of December 31, 2011 using each of the following three methods:

- ◇ Chain ladder
- ◇ Bornhuetter/Ferguson
- ◇ Benktander

b) Determine the accident year 2011 incremental paid loss in 2012 that would result in the Benktander ultimate loss estimate being \$50,000 greater than the Bornhuetter/Ferguson ultimate loss estimate for accident year 2011, as of December 31, 2012. Assume all selected development factors remain the same.

Solution to part a:

- ◇ Chain-ladder
  - $U_{CL} = 700000(2.5) = \boxed{\$1{,}750{,}000}$
- ◇ Bornhuetter/Ferguson
  - $U_{BF} = C_k + q_k U_0 = 700000 + (1 - 1/2.5)(3000000)(0.625) = \boxed{\$1{,}825{,}000}$
- ◇ Benktander
  - $U_{GB} = C_k + q_k U_{BF} = 7000000 + (1 - 1/2.5)(1825000) = \boxed{\$1{,}795{,}000}$

Solution to part b:

- ◇ $U_{GB} = U_{BF} + 50000$
- ◇ $C_k + q_k U_{BF} = U_{BF} + 50000$
- ◇ $C_k - 50000 = U_{BF}(1 - q_k)$
- ◇ Let the incremental paid loss in 2012 for AY 2011 be x
- ◇ $700000 + x - 50000 = U_{BF}(1 - q_k)$
- ◇ $650000 + x = U_{BF}(p_k)$
- ◇ $650000 + x = U_{BF}\left(\frac{1}{LDF_{24-ult}}\right)$
- ◇ $650000 + x = U_{BF}\left(\frac{1}{2.5/1.5}\right)$
- ◇ $650000 + x = U_{BF}(0.6)$
- ◇ $650000 + x = (C_k + q_k U_0)(0.6)$
- ◇ $650000 + x = (700000 + x + 0.4(3000000)(0.625))(0.6)$
- ◇ $650000 + x = 870000 + 0.6x$
- ◇ $0.4x = 220000$
- ◇ $x = \boxed{\$550{,}000}$

Hürlimann

Outline

I. Introduction

- Hürlimann's method is inspired by the Benktander method
- A couple of differences between Hürlimann's method and the Benktander method:
  - Hürlimann's method is based on a full development triangle, whereas the Benktander method is based on a single origin period (i.e. accident year or underwriting year)
  - Hürlimann's method requires a measure of exposure for each origin period (i.e. premiums)
- Unlike standard reserving methods that rely on link ratios to determine reserves (chain-ladder, Bornhuetter/Ferguson, Cape Cod), Hürlimann's method relies on loss ratios
- The **main result of the method** is that it provides an optimal credibility weight for combining the chain-ladder or individual loss ratio reserve (grossed up latest claims experience of an origin period) with the Bornhuetter/Ferguson or collective loss ratio reserve (experience based burning cost estimate of the total ultimate claims of an origin period)

II. The Collective and Individual Loss Ratio Claims Reserves

- Notation
  - p_i is the proportion of the total ultimate claims from origin period i expected to be paid in development period $n-i+1$ (known as the loss ratio payout factor or loss ratio lag-factor)
  - $q_i = 1 - p_i$ is the proportion of the total ultimate claims from origin period i which remain unpaid in development period $n-i+1$ (known as the loss ratio reserve factor)
  - $U_i^{BC} = U_i^{(0)}$ is the burning cost of the total ultimate claims for origin period i
  - $U_i^{coll} = U_i^{(1)}$ is the collective total ultimate claims for origin period i
  - $U_i^{ind} = U_i^{(\infty)}$ is the individual total ultimate claims for origin period i
  - $U_i^{(m)}$ is the ultimate claim estimate at the m^{th} iteration for origin period i
  - R_i^{coll} is the collective loss ratio claims reserve for origin period i
  - R_i^{ind} is the individual loss ratio claims reserve for origin period i

- R_i^c is the credible loss ratio claims reserve
- R_i^{GB} is the Benktander loss ratio claims reserve
- R_i^{WN} is the Neuhaus loss ratio claims reserve
- R_i is the i-th period claims reserve for origin period i
- R is the total claims reserve
- m_k is the expected loss ratio in development period k
- n is the number of origin periods
- V_i is the premium belonging to origin period i
- S_{ik} are the paid claims from origin period i as of k years of development where $1 \leq i, k \leq n$
- C_{ik} are the cumulative paid claims from origin period i as of k years of development

◇ Assuming that after n development periods all claims incurred in an origin period are known and closed, the **total ultimate claims** from origin period i are:

$$\boxed{\sum_{k=1}^{n} S_{ik}}$$

◇ **Cumulative paid claims**

$$\boxed{C_{ik} = \sum_{j=1}^{k} S_{ij}}$$

◇ ***i*-th period claims reserve**

- The required amount for the incurred but unpaid claims of origin period i

$$\boxed{R_i = \sum_{k=n-i+2}^{n} S_{ik}}$$

where $i = 2, ..., n$

◇ **Total claims reserve**

- The total amount of incurred but unpaid claims over all periods

$$R = \sum_{i=2}^{n} R_i$$

◇ **Expected loss ratio**

- The incremental amount of expected paid claims per unit of premium in each development period (i.e. an incremental loss ratio)

$$m_k = \frac{E\left[\sum_{i=1}^{n-k+1} S_{ik}\right]}{\sum_{i=1}^{n-k+1} V_i}$$

where $k = 1, ..., n$

◇ **Expected value of the burning cost** of the total ultimate claims

- This quantity is similar to the prior estimate U_0 from Mack (2000)

$$E\left[U_i^{BC}\right] = V_i \cdot \sum_{k=1}^{n} m_k$$

- By summing up the m_k's (the incremental loss ratios), we obtain an overall expected loss ratio. When we multiply the overall expected loss ratio by the premium V_i, we obtain an expected loss for each origin period

◇ **Loss ratio payout factor**

- Represents the percent of losses emerged to date for each origin period

$$p_i = \frac{V_i \cdot \sum_{k=1}^{n-i+1} m_k}{E[U_i^{BC}]} = \frac{\sum_{k=1}^{n-i+1} m_k}{\sum_{k=1}^{n} m_k}$$

◇ **Individual total ultimate claims**

- Obtained by grossing up the latest cumulative paid claims for an origin period
- Considered "individual" since it depends on the individual latest claims experience of an origin period

- This estimate is similar to the chain-ladder (CL) estimate from Mack (2000)

$$\boxed{U_i^{ind} = \frac{C_{i,n-i+1}}{p_i}}$$

◇ **Individual loss ratio claims reserve**

$$\boxed{\begin{aligned} R_i^{ind} &= U_i^{ind} - C_{i,n-i+1} \\ &= q_i \cdot U_i^{ind} \\ &= \frac{q_i}{p_i} \cdot C_{i,n-i+1} \end{aligned}}$$

◇ **Collective loss ratio claims reserve**

- Obtained by using the burning cost of the total ultimate claims
- Considered "collective" since it depends on the portfolio claims experience of all origin periods

$$\boxed{R_i^{coll} = q_i \cdot U_i^{BC}}$$

◇ **Collective total ultimate claims**

- This estimate is similar to the Bornhuetter/Ferguson (BF) estimate from Mack (2000)

$$\boxed{U_i^{coll} = R_i^{coll} + C_{i,n-i+1}}$$

◇ An **advantage** of the collective loss ratio claims reserve over the BF reserve is that different actuaries always come to the same results provided they use the same premiums

III. Credible Loss Ratio Claims Reserve

◇ The individual and collective loss ratio claims reserve estimates represent extreme positions

- The individual claims reserve assumes that the cumulative paid claims amount $C_{i,n-i+1}$ is fully credible for future claims and ignores the burning cost U_i^{BC} of the total ultimate claims
- The collective claims reserve ignores the cumulative paid claims and relies fully on the burning cost

◇ **Credible loss ratio claims reserve**

- Mixture of the individual and collective loss ratio reserves

$$\boxed{R_i^c = Z_i \cdot R_i^{ind} + (1 - Z_i) \cdot R_i^{coll}}$$

where Z_i is the credibility weight given to the individual loss ratio reserve

- **Benktander loss ratio claims reserve**
  - Obtained by setting $Z_i = Z_i^{GB} = p_i$

$$\boxed{R_i^{GB} = p_i \cdot R_i^{ind} + q_i \cdot R_i^{coll}}$$

- **Neuhaus loss ratio claims reserve**
  - Obtained by setting $Z_i = Z_i^{WN} = \sum_{k=1}^{n-i+1} m_k = p_i \cdot \sum_{k=1}^{n} m_k$

$$\boxed{R_i^{WN} = Z_i^{WN} \cdot R_i^{ind} + (1 - Z_i^{WN}) \cdot R_i^{coll}}$$

- At this point in the paper, Hürlimann restates the theorem from Mack (2000) that shows how ultimates and reserves change as we iterate between them
- Using the iteration rules $R_i^{(m)} = q_i U_i^{(m)}$ and $U_i^{(m+1)} = C_{i,n-i+1} + q_i U_i^{(m)}$, we obtain the following credibility mixtures:

$$\boxed{\begin{aligned} U_i^{(m)} &= (1 - q_i^m) U_i^{ind} + q_i^m U_i^0 \\ R_i^{(m)} &= (1 - q_i^m) R_i^{ind} + q_i^m R_i^0 \end{aligned}}$$

- Once again, if we iterate between reserves and ultimates indefinitely, we eventually end up with the individual loss ratio estimate for ultimate claims.

IV. The Optimal Credibility Weights and the Mean Squared Error

- The optimal credibility weights Z_i^* which minimize the mean squared error $\text{mse}(R_i^c) = E[(R_i^c - R_i)^2]$ are given by:

$$\boxed{Z_i^* = \frac{p_i}{p_i + t_i}}$$

  where $t_i = \frac{E[\alpha_i^2(U_i)]}{Var(U_i^{BC}) + Var(U_i) - E[\alpha_i^2(U_i)]}$

- *In the paper, the author goes into quite a bit of detail on how to estimate the quantities in the formula for t_i above. I believe that these details are outside of the scope of the exam and are excluded from this outline*
- The weights Z_i^* which minimize the mean squared error $\text{mse}(R_i^c) = E[(R_i^c - R_i)^2]$ and the variance $Var(R_i^c)$ are obtained by:

$$\boxed{t_i^* = \frac{f_i - 1 + \sqrt{(f_i + 1) \cdot (f_i - 1 + 2p_i)}}{2}}$$

- *Note that f_i comes from an assumption the author makes in the paper. He assumes that U_i is at least as volatile as the burning cost estimate U_i^{BC}. Thus, $Var(U_i) = f_i \cdot Var(U_i^{BC})$*

- A special case of the formula above is when $f_i = 1$. This implies that $Var(U_i) = Var(U_i^{BC})$. In this case, t_i can be estimated by

$$\boxed{t_i^* = \sqrt{p_i}}$$

  This is the case I expect to see on the exam. Thus, unless told otherwise, assume that $t_i = t_i^* = \sqrt{p_i}$. *Note that the online CAS text references provide two different versions of this paper. Each version of the paper has a different version of the formula above. If you navigate to the online text references and click on the first link under Hürlimann, you will find that $t_i^* = \sqrt{p_i}$. If you download the "complete PDF of online text references," it provides the second version of this paper with a different formula for t_i^*. Given that $t_i^* = \sqrt{p_i}$ is what is shown in all of the solutions on prior exams, I recommend using this version of the formula*

- Since $t_i^* = \sqrt{p_i} \leq 1$, $Z_i^* \leq \frac{1}{2}$

- **According to the author**, this special case is appealing because it yields the smallest credibility weights for the individual loss reserves, which places more emphasis on the collective loss reserves (*I say "According to the author" because this is not correct. As f increases from $f = 1$, the credibility Z actually decreases, placing less weight on the individual loss reserves. If this comes up as a short answer question on the exam, stick with what the author says*)

- The mean squared error for the credible loss ratio reserve is given by:

$$\boxed{\text{mse}(R_i^c) = E[\alpha_i^2(U_i)] \cdot \left[\frac{Z_i^2}{p_i} + \frac{1}{q_i} + \frac{(1-Z_i)^2}{t_i}\right] \cdot q_i^2}$$

- The mean squared errors for the collective and individual loss ratios reserves can be obtained by setting Z_i equal to 0 and 1, respectively

V. Example

- Given the following incremental losses:

		Dev. Period		
i	V_i = Premium	1	2	3
1	15	10	4	2
2	20	6	5	
3	22	8		

◇ Calculate the following parameters:

i or k	m_k	$p_i = Z_i^{GB}$	q_i	t_i^*	Z_i^*	Z_i^{WN}
1	0.421	1.000	0.000	1.000	0.500	0.811
2	0.257	0.836	0.164	0.914	0.478	0.678
3	0.133	0.519	0.481	0.720	0.419	0.421

◇ Here are the underlying calculations:

- $m_k = \frac{E\left[\sum_{i=1}^{n-k+1} S_{ik}\right]}{\sum_{i=1}^{n-k+1} V_i}$
  - ◇ $m_1 = \frac{10+6+8}{15+20+22} = 0.421$
  - ◇ $m_2 = \frac{4+5}{15+20} = 0.257$
  - ◇ $m_3 = \frac{2}{15} = 0.133$
- $p_i = \frac{\sum_{k=1}^{n-i+1} m_k}{\sum_{k=1}^{n} m_k}$
  - ◇ $p_1 = \frac{0.421+0.257+0.133}{0.421+0.257+0.133} = 1.000$
  - ◇ $p_2 = \frac{0.421+0.257}{0.421+0.257+0.133} = 0.836$
  - ◇ $p_3 = \frac{0.421}{0.421+0.257+0.133} = 0.519$
- $q_i = 1 - p_i$
  - ◇ $q_1 = 1 - 1 = 0.000$
  - ◇ $q_2 = 1 - 0.836 = 0.164$
  - ◇ $q_3 = 1 - 0.519 = 0.481$
- $t_i^* = \sqrt{p_i}$ (assumes that $Var(U_i) = Var(U_i^{BC})$)
  - ◇ $t_1^* = \sqrt{1} = 1.000$
  - ◇ $t_2^* = \sqrt{0.836} = 0.914$
  - ◇ $t_3^* = \sqrt{0.519} = 0.720$
- $Z_i^* = \frac{p_i}{p_i + t_i^*}$
  - ◇ $Z_1^* = \frac{1}{1+1} = 0.500$
  - ◇ $Z_2^* = \frac{0.836}{0.836+0.914} = 0.478$
  - ◇ $Z_3^* = \frac{0.519}{0.519+0.720} = 0.419$

- $Z_i^{WN} = \sum_{k=1}^{n-i+1} m_k$
  - ◇ $Z_1^{WN} = 0.421 + 0.257 + 0.133 = 0.811$
  - ◇ $Z_2^{WN} = 0.421 + 0.257 = 0.678$
  - ◇ $Z_3^{WN} = 0.421$

◇ Calculate the reserves:

i	Collective	Individual	Neuhaus	Benktander	Optimal
2	2.660	2.158	2.320	2.240	2.420
3	8.582	7.414	8.090	7.976	8.093

◇ Here are the underlying calculations for the collective, individual, and Neuhaus reserves for origin period 2:

- Collective $= q_i \cdot U_i^{BC} = 0.164(20)(0.421 + 0.257 + 0.133) = 2.660$ (similar to BF)
- Individual $= \frac{C_{i,n-i+1}}{p_i} - C_{i,n-i+1} = \frac{6+5}{0.836} - (6+5) = 2.158$ (similar to CL)
- Neuhaus $= Z_i^{WN} \cdot R_i^{ind} + (1 - Z_i^{WN}) \cdot R_i^{coll} = 0.678(2.158) + (1 - 0.678)(2.660) = 2.320$

◇ Calculate the relative MSE's for each method (i.e. divide each method's MSE by the optimal MSE):

i	Collective	Individual	Neuhaus	Benktander	Optimal
2	1.078	1.094	1.014	1.044	1.000
3	1.202	1.388	1.000	1.012	1.000

◇ Here are the underlying calculations for the collective, individual, and Neuhaus reserves for origin period 2:

- Collective $= \dfrac{E[\alpha_i^2(U_i)] \cdot \left[\frac{0^2}{0.836} + \frac{1}{0.164} + \frac{(1-0)^2}{0.914}\right] \cdot 0.164^2}{E[\alpha_i^2(U_i)] \cdot \left[\frac{0.478^2}{0.836} + \frac{1}{0.164} + \frac{(1-0.478)^2}{0.914}\right] \cdot 0.164^2} = 1.078$
- Individual $= \dfrac{E[\alpha_i^2(U_i)] \cdot \left[\frac{1^2}{0.836} + \frac{1}{0.164} + \frac{(1-1)^2}{0.914}\right] \cdot 0.164^2}{E[\alpha_i^2(U_i)] \cdot \left[\frac{0.478^2}{0.836} + \frac{1}{0.164} + \frac{(1-0.478)^2}{0.914}\right] \cdot 0.164^2} = 1.094$
- Neuhaus $= \dfrac{E[\alpha_i^2(U_i)] \cdot \left[\frac{0.678^2}{0.836} + \frac{1}{0.164} + \frac{(1-0.678)^2}{0.914}\right] \cdot 0.164^2}{E[\alpha_i^2(U_i)] \cdot \left[\frac{0.478^2}{0.836} + \frac{1}{0.164} + \frac{(1-0.478)^2}{0.914}\right] \cdot 0.164^2} = 1.014$

◇ Using the relative MSE table, it's clear that the Neuhaus reserve best matches the optimal credible reserve

VI. Reinterpreting the Methods from Mack (2000)

◇ *Note: In this section, the author is making connections between this paper and the Mack (2000) paper. Thus, we are using the standard age-to-age factors in this section*

◇ Let $f_k^{CL} = \frac{\sum_{i=1}^{n-k} C_{i,k+1}}{\sum_{i=1}^{n-k} C_{ik}}$. These are the chain-ladder age-to-age factors

◇ Let $F_k^{CL} = \prod_{j=k}^{n-1} f_j^{CL}$. These are the chain-ladder age-to-ultimate factors

◇ Let $p_i^{CL} = \frac{1}{F_{n-i+1}^{CL}}$. These are the chain-ladder lag-factors

◇ Let $q_i^{CL} = 1 - p_i^{CL}$. These are the chain-ladder reserve factors

◇ **Chain-ladder method**

- This is the individual loss ratio method with loss ratio lag-factors replaced by the chain-ladder lag-factors:

$$\boxed{R_i^{CL} = \frac{q_i^{CL}}{p_i^{CL}} \cdot C_{i,n-i+1}}$$

◇ **Cape Cod method**

- Benktander-type credibility mixture with the following components:

$$\boxed{\begin{aligned} R_i^{\text{ind}} &= \frac{q_i^{CL}}{p_i^{CL}} \cdot C_{i,n-i+1} \\ R_i^{\text{coll}} &= q_i^{CL} \cdot LR \cdot V_i \\ Z_i &= p_i^{CL} \end{aligned}}$$

where $LR = \frac{\sum_{i=1}^{n} C_{i,n-i+1}}{\sum_{i=1}^{n} p_i^{CL} \cdot V_i}$

- *Note: The credibility mixture above does not equal the Cape Cod method. Instead, the collective reserves defined above equal the standard Cape Cod reserves. Thus, the credibility estimate is mixture of the chain-ladder reserve estimate and the standard Cape Cod reserve estimate*

◇ **Optimal Cape Cod method**

- Identical to the Cape Cod method, but with the following credibility weights:

$$\boxed{Z_i = \frac{p_i^{CL}}{p_i^{CL} + \sqrt{p_i^{CL}}}}$$

◇ **Bornhuetter/Ferguson method**

- Benktander-type credibility mixture with the following components:

$$\boxed{\begin{aligned} R_i^{\text{ind}} &= \frac{q_i^{CL}}{p_i^{CL}} \cdot C_{i,n-i+1} \\ R_i^{\text{coll}} &= q_i^{CL} \cdot LR_i \cdot V_i \\ Z_i &= p_i^{CL} \end{aligned}}$$

  where LR_i is some selected initial loss ratio for each origin period

- *Note: The credibility mixture above does not equal the BF method. Instead, the collective reserves defined above equal the standard BF reserves. Thus, the credibility estimate is mixture of the chain-ladder reserve estimate and the standard BF reserve estimate*

◇ **Optimal Bornhuetter/Ferguson method**

- Identical to the Bornhuetter/Ferguson method, but with the following credibility weights:

$$\boxed{Z_i = \frac{p_i^{CL}}{p_i^{CL} + \sqrt{p_i^{CL}}}}$$

Original Mathematical Problems & Solutions

MP #1

Given the following:

- $U_2^{ind} = \$5{,}000$
- $C_{2,3} = \$4{,}500$
- $q_2 = 0.10$
- $n = 4$

Calculate R_2^{ind} in three different ways.

Solution:

- ◇ Method 1:
  - $R_2^{ind} = U_2^{ind} - C_{2,3} = 5000 - 4500 = \boxed{\$500}$
- ◇ Method 2:
  - $R_2^{ind} = q_2 \cdot U_2^{ind} = 0.10(5000) = \boxed{\$500}$
- ◇ Method 3:
  - $R_2^{ind} = \frac{q_2 \cdot C_{2,3}}{1-q_2} = \frac{0.10(4500)}{1-0.10} = \boxed{\$500}$

MP #2

Given the following:

		Incremental Incurred Losses ($)			
AY	Earned Premium($)	12 mo.	24 mo.	36 mo.	48 mo.
2009	7,000	4,000	2,000	500	200
2010	7,500	3,000	2,500	600	
2011	8,000	4,500	1,500		
2012	8,500	5,000			

a) Estimate the AY 2011 ultimate losses using the collective loss ratio method.

b) Estimate the AY 2011 ultimate losses using the individual loss ratio method.

c) Estimate the AY 2011 ultimate losses using the Neuhaus method.

d) Estimate the AY 2011 ultimate losses using the Benktander method.

e) Estimate the AY 2011 ultimate losses using the optimal credibility weights that minimize the variance of the credible claims reserve. Assume that $Var(U_i) = Var(U_i^{BC})$.

f) Use relative MSE's to explain which method in parts a. - d. best matches the optimal reserve calculated in part e.

Solution to part a:

- ⋄ Calculate the m_k's:
  - We know that $m_k = \frac{E\left[\sum_{i=1}^{n-k+1} S_{ik}\right]}{\sum_{i=1}^{n-k+1} V_i}$
  - Thus, we can create the following table:

k	m_k
1	$0.532 = \frac{4000+3000+4500+5000}{7000+7500+8000+8500}$
2	$0.267 = \frac{2000+2500+1500}{7000+7500+8000}$
3	0.076
4	0.029

- ⋄ Calculate $E[U_3^{BC}]$:
  - We know that $E\left[U_i^{BC}\right] = V_i \cdot \sum_{k=1}^{n} m_k$
  - Thus, $E\left[U_3^{BC}\right] = 8000(0.532 + 0.267 + 0.076 + 0.029) = 7232$
- ⋄ Calculate R_3^{coll}:
  - We know that $R_i^{coll} = q_i \cdot U_i^{BC}$
  - $p_i = \frac{\sum_{k=1}^{n-i+1} m_k}{\sum_{k=1}^{n} m_k}$
  - Thus, $p_3 = \frac{0.532+0.267}{0.532+0.267+0.076+0.029} = 0.884$ and $q_3 = 1 - p_3 = 0.116$
  - Thus, $R_3^{coll} = q_3 \cdot U_3^{BC} = 0.116(7232) = 838.912$
- ⋄ Calculate U_3^{coll}:
  - $U_3^{coll} = R_3^{coll} + C_{3,2} = 838.912 + (4500 + 1500) = \boxed{\$6{,}838.91}$

Solution to part b:

- ⋄ Calculate R_3^{ind}:
  - We know that $R_i^{ind} = \frac{q_i}{p_i} \cdot C_{i,n-i+1}$
  - Thus, $R_3^{ind} = \frac{q_3}{p_3} \cdot C_{3,2} = \frac{0.116}{0.884}(4500 + 1500) = 787.33$
- ⋄ Calculate U_3^{ind}:
  - $U_3^{ind} = R_3^{ind} + C_{3,2} = 787.33 + (4500 + 1500) = \boxed{\$6{,}787.33}$

Solution to part c:

◇ Calculate Z_3^{WN}:

- We know that $Z_i^{WN} = \sum_{k=1}^{n-i+1} m_k$
- Thus, $Z_3^{WN} = 0.532 + 0.267 = 0.799$

◇ Calculate R_3^{WN}:

- We know that $R_i^{WN} = Z_i^{WN} \cdot R_i^{ind} + (1 - Z_i^{WN}) \cdot R_i^{coll}$
- Thus, $R_3^{WN} = Z_3^{WN} \cdot R_3^{ind} + (1 - Z_3^{WN}) \cdot R_3^{coll} = 0.799(787.33) + (1 - 0.799)(838.912) = 797.698$

◇ Calculate U_3^{WN}:

- $U_3^{WN} = R_3^{WN} + C_{3,2} = 797.698 + (4500 + 1500) = \boxed{\$6{,}797.70}$

Solution to part d:

◇ Calculate R_3^{GB}:

- We know that $R_i^{GB} = p_i \cdot R_i^{ind} + q_i \cdot R_i^{coll}$
- Thus, $R_3^{GB} = p_3 \cdot R_3^{ind} + q_3 \cdot R_3^{coll} = 0.884(787.33) + 0.116(838.912) = 793.314$

◇ Calculate U_3^{GB}:

- $U_3^{GB} = R_3^{GB} + C_{3,2} = 793.314 + (4500 + 1500) = \boxed{\$6{,}793.31}$

Solution to part e:

◇ Calculate Z_i^*:

- We know that $Z_i^* = \frac{p_i}{p_i + t_i}$
- Thus, $Z_3^* = \frac{p_3}{p_3 + t_3} = \frac{0.884}{0.884 + \sqrt{0.884}} = 0.485$

◇ Calculate the optimal reserves (call these R_3^{opt}):

- We know that $R_i^c = Z_i \cdot R_i^{ind} + (1 - Z_i) \cdot R_i^{coll}$
- Thus, $R_3^{opt} = Z_3^* \cdot R_3^{ind} + (1 - Z_3^*) \cdot R_3^{coll} = 0.485(787.33) + (1 - 0.485)(838.912) = 813.895$

◇ Calculate the optimal ultimate losses (call these U_3^{opt}):

- $U_3^{opt} = R_3^{opt} + C_{3,2} = 813.895 + (4500 + 1500) = \boxed{\$6{,}813.90}$

Solution to part f:

◇ Calculate the relative MSE's for each method (i.e. divide each method's MSE by the optimal MSE):

i	Collective	Individual	Neuhaus	Benktander	Optimal
3	1.056	1.064	1.024	1.038	1.000

◇ Here are the underlying calculations:

- Collective $= \dfrac{E[\alpha_i^2(U_i)]\cdot\left[\frac{0^2}{0.884}+\frac{1}{0.116}+\frac{(1-0)^2}{0.940}\right]\cdot 0.116^2}{E[\alpha_i^2(U_i)]\cdot\left[\frac{0.485^2}{0.884}+\frac{1}{0.116}+\frac{(1-0.485)^2}{0.940}\right]\cdot 0.116^2} = 1.056$
- Individual $= \dfrac{E[\alpha_i^2(U_i)]\cdot\left[\frac{1^2}{0.884}+\frac{1}{0.116}+\frac{(1-1)^2}{0.940}\right]\cdot 0.116^2}{E[\alpha_i^2(U_i)]\cdot\left[\frac{0.485^2}{0.884}+\frac{1}{0.116}+\frac{(1-0.485)^2}{0.940}\right]\cdot 0.116^2} = 1.064$
- Neuhaus $= \dfrac{E[\alpha_i^2(U_i)]\cdot\left[\frac{0.799^2}{0.884}+\frac{1}{0.116}+\frac{(1-0.799)^2}{0.940}\right]\cdot 0.116^2}{E[\alpha_i^2(U_i)]\cdot\left[\frac{0.485^2}{0.884}+\frac{1}{0.116}+\frac{(1-0.485)^2}{0.940}\right]\cdot 0.116^2} = 1.024$
- Benktander $= \dfrac{E[\alpha_i^2(U_i)]\cdot\left[\frac{0.884^2}{0.884}+\frac{1}{0.116}+\frac{(1-0.884)^2}{0.940}\right]\cdot 0.116^2}{E[\alpha_i^2(U_i)]\cdot\left[\frac{0.485^2}{0.884}+\frac{1}{0.116}+\frac{(1-0.485)^2}{0.940}\right]\cdot 0.116^2} = 1.038$

◇ Using the relative MSE table, it's clear that the **Neuhaus reserve** best matches the optimal credible reserve

MP #3

Given the following for a 4 x 4 triangle:

- ◇ $U_4^{(0)} = \$5{,}000$
- ◇ $C_{4,1} = \$1{,}200$
- ◇ $q_4 = 0.80$

Calculate $U_4^{(3)}$.

Solution:

- $\diamond$ $R_4^{(0)} = q_4 \cdot U_4^{(0)} = 0.8(5000) = 4000$
- $\diamond$ $U_4^{(1)} = C_{4,1} + R_4^{(0)} = 1200 + 4000 = 5200$
- $\diamond$ $R_4^{(1)} = q_4 \cdot U_4^{(1)} = 0.8(5200) = 4160$
- $\diamond$ $U_4^{(2)} = C_{4,1} + R_4^{(1)} = 1200 + 4160 = 5360$
- $\diamond$ $R_4^{(2)} = q_4 \cdot U_4^{(2)} = 0.8(5360) = 4288$
- $\diamond$ $U_4^{(3)} = C_{4,1} + R_4^{(2)} = 1200 + 4288 = \boxed{\$5{,}488}$

MP #4

Given the following:

- ⋄ $f_2 = 1.3$
- ⋄ $p_2 = 0.9$
- ⋄ $R_2^{\text{ind}} = \$5{,}000$
- ⋄ $R_2^{\text{coll}} = \$4{,}500$

Using credibility weights that minimize the variance of the optimal credibility claims reserve, estimate R_2^c.

Solution:

◇ Calculate t_2^*:

- $t_2^* = \frac{f_2 - 1 + \sqrt{(f_2+1)\cdot(f_2-1+2p_2)}}{2} = \frac{1.3-1+\sqrt{(1.3+1)\cdot(1.3-1+2(0.9))}}{2} = 1.249$

◇ Calculate Z_2^*:

- $Z_2^* = \frac{p_2}{p_2+t_2^*} = \frac{0.9}{0.9+1.249} = 0.419$

◇ Calculate R_2^c:

- $R_2^c = R_2^{\text{ind}} \cdot Z_2^* + R_2^{\text{coll}} \cdot (1 - Z_2^*) = 5000(0.419) + (1-0.419)(4500) = \boxed{\$4{,}709.50}$

MP #5

Given the following:

- ◇ $f_2 = 1$
- ◇ $t_2^* = 0.95$
- ◇ Individual loss ratio claims reserve = $5,000
- ◇ Minimum variance claims reserve = $4,800

Calculate the collective loss ratio claims reserve for origin period 2.

Solution:

◇ Calculate Z_2^*:

- Since $f_2 = 1$, $t_2^* = 0.95 = \sqrt{p_2}$. Thus, $p_2 = 0.903$
- $Z_2^* = \frac{p_2}{p_2+t_2^*} = \frac{0.903}{0.903+0.95} = 0.487$

◇ Calculate R_2^{coll}:

- $R_2^{\text{c}} = R_2^{\text{ind}} \cdot Z_2^* + R_2^{\text{coll}} \cdot (1 - Z_2^*)$
- $4800 = 5000(0.487) + (1 - 0.487) \cdot R_2^{\text{coll}}$
- Thus, $R_2^{\text{coll}} = \boxed{\$4{,}610.14}$

MP #6

Given the following:

AY	Earned Premium($)	Cumulative Reported Losses ($)		
		12 mo.	24 mo.	36 mo.
2010	200	40	80	100
2011	225	60	120	
2012	250	65		

a) Estimate the AY 2012 reserves using the optimal Cape Cod method.

b) Estimate the AY 2012 reserves using the optimal Bornhuetter/Ferguson method given an initial loss ratio of 0.55.

Solution to part a:

◇ Calculate the age-to-age factors:

- $f_1^{CL} = \frac{80+120}{40+60} = 2$
- $f_2^{CL} = \frac{100}{80} = 1.25$

◇ Calculate the p_i^{CL}'s:

- $p_1^{CL} = 1$
- $p_2^{CL} = \frac{1}{1.25} = 0.80$
- $p_3^{CL} = \frac{1}{2(1.25)} = 0.40$

◇ Calculate R_3^{ind}:

- $R_3^{\text{ind}} = \frac{q_3^{CL}}{p_3^{CL}} \cdot C_{3,1} = \frac{1-0.40}{0.40} \cdot 65 = 97.5$

◇ Calculate R_3^{coll}:

- $R_3^{\text{coll}} = V_3 \cdot LR \cdot q_3$
- $LR = \frac{\sum_{i=1}^{n} C_{i,n-i+1}}{\sum_{i=1}^{n} p_i^{CL} \cdot V_i} = \frac{100+120+65}{200(1)+225(0.80)+250(0.40)} = 0.594$
- Thus, $R_3^{\text{coll}} = 250(0.594)(1 - 0.40) = 89.1$

◇ Calculate Z_3^*:

- $Z_3^* = \frac{p_3^{CL}}{p_3^{CL} + \sqrt{p_3^{CL}}} = \frac{0.40}{0.40+\sqrt{0.40}} = 0.387$

◇ Calculate R_3^c:

- $R_3^c = 97.5(0.387) + (1 - 0.387)(89.1) = \boxed{\$92.35}$

Solution to part b:

◇ Calculate R_3^{coll}:

- $R_3^{\text{coll}} = V_3 \cdot LR_3 \cdot q_3 = 250(0.55)(1 - 0.40) = 82.5$

◇ Calculate R_3^c:

- $R_3^c = 97.5(0.387) + (1 - 0.387)(82.5) = \boxed{\$88.31}$

Original Essay Problems

EP #1

a) Briefly describe three differences between Hürlimann's method and the Benktander method.

b) Briefly describe one similarity between Hürlimann's method and the Benktander method.

EP #2

Provide one advantage of the collective loss ratio reserve over the standard Bornhuetter/Ferguson reserve.

EP #3

Explain why $t_i^* = \sqrt{p_i}$ is an appealing choice when calculating the optimal credibility weights.

Original Essay Solutions

ES #1

Part a:

- ◇ Hürlimann's method is based on a full development triangle, whereas the Benktander method is based on a single accident year
- ◇ Hürlimann's method requires a measure of exposure for each accident year (i.e. premiums)
- ◇ Hürlimann's method relies on loss ratios (rather than link ratios) to determine reserves

Part b:

- ◇ Similar to the Benktander method, Hürlimann's method represents a credibility weighting between two extreme positions: relies on cumulative paid claims (i.e. individual loss reserves) vs. ignores cumulative paid claims (i.e. collective loss reserves)

ES #2

- ◇ With the collective loss ratio reserve, different actuaries always come to the same results provided they use the same premiums

ES #3

- ◇ This assumption yields the smallest credibility weights for the individual loss reserves, which places more emphasis on the collective loss reserves (*as mentioned in the outline, this does not appear to be correct. As* f *increases from* $f = 1$*, less weight is placed on the individual loss reserves. That being said, I think there's a possibility this could be asked on the exam. If so, stick with what the author says*)

Past CAS Exam Problems & Solutions

2019 #2

Given the following information as of December 31, 2018:

AY	Earned Premium ($000)	Inc. Paid Loss ($000) as of (months) 12	24	36
2016	5,000	1,800	700	500
2017	6,000	2,000	800	
2018	8,000	2,200		

- ◇ Assume there is no further development after 36 months
- ◇ $Var(U_i) = Var(U_i^{BC})$

a) Calculate the accident year 2018 Benktander reserve estimate (R^{GB}).

b) Calculate the accident year 2018 optimal credible reserve estimate (R_c).

c) Identify which of R_c or R^{GB} is the preferable reserve from a statistical point of view and briefly describe a supporting reason.

d) Describe the effect on the Benktander credibility for accident year 2018 if the incremental paid loss from 12 to 24 months for accident year 2017 was greater than the value in the table above.

Solution to part a:

◇ Calculate the m_k's:

- $m_k = \frac{E\left[\sum_{i=1}^{n-k+1} S_{ik}\right]}{\sum_{i=1}^{n-k+1} V_i}$
- $m_1 = 0.316 = \frac{1800+2000+2200}{5000+6000+8000}$
- $m_2 = 0.136$
- $m_3 = 0.100$

◇ Calculate $E[U_3^{BC}]$:

- $E\left[U_i^{BC}\right] = V_i \cdot \sum_{k=1}^{n} m_k$
- $E\left[U_3^{BC}\right] = 8000(0.316 + 0.136 + 0.100) = 4416$

◇ Calculate p_3 and q_3:

- $p_i = \frac{\sum_{k=1}^{n-i+1} m_k}{\sum_{k=1}^{n} m_k}$
- $p_3 = \frac{0.316}{0.316+0.136+0.100} = 0.572$ and $q_3 = 1 - p_3 = 0.428$

◇ Calculate R_3^{ind}:

- $R_i^{ind} = \frac{q_i}{p_i} \cdot C_{i,n-i+1}$ and $U_i^{ind} = R_i^{ind} + C_{i,n-i+1}$
- $R_3^{ind} = \frac{0.428}{0.572} \cdot 2200 = 1646.154$

◇ Calculate R_3^{coll}:

- $R_i^{coll} = q_i \cdot U_i^{BC}$ and $U_i^{coll} = R_i^{coll} + C_{i,n-i+1}$
- $R_3^{coll} = 0.428(4416) = 1890.048$

◇ Calculate R_3^{GB}:

- $R_i^{GB} = Z_i^{GB} \cdot R_i^{ind} + (1 - Z_i^{GB}) \cdot R_i^{coll}$, where $Z_i^{GB} = p_i$
- $R_3^{GB} = p_3 \cdot R_3^{ind} + (1-p_3) \cdot R_3^{coll} = 0.572(1646.154) + (1-0.572)(1890.048) = \boxed{\$1,750,541}$

Solution to part b:

◇ Since $Var(U_i) = Var(U_i^{BC})$, $Z_3^c = \frac{p_3}{p_3+\sqrt{p_3}} = \frac{0.572}{0.572+\sqrt{0.572}} = 0.431$

◇ $R_3^c = Z_3^c \cdot R_3^{ind} + (1 - Z_3^c) \cdot R_3^{coll} = 0.431(1646.154) + (1 - 0.431)(1890.048) = \boxed{\$1,784,930}$

Solution to part c:

- ◇ R_c is preferable because it minimizes the MSE of the reserve

Solution to part d:

- ◇ In this case, m_2 would increase, while m_1 and m_3 would remain the same. Thus, $p_3 = \frac{m_1}{m_1+m_2+m_3}$ would decrease since the denominator increases while the numerator stays the same. Since $Z_3^{GB} = p_3$, the credibility decreases

2019 #3

Given the following information as of December 31, 2018:

		Inc. Paid Loss ($000) as of (months)		
AY	Earned Premium ($000)	12	24	36
2016	800	320	220	80
2017	600	300	200	
2018	400	280		

- ◇ Assume there is no loss development beyond 36 months

a) Calculate the total Neuhaus loss ratio claims reserve estimate.

b) Describe why the Neuhaus method may not be appropriate for the data in the table above.

Solution to part a:

◇ Calculate the m_k's:

- $m_k = \frac{E\left[\sum\limits_{i=1}^{n-k+1} S_{ik}\right]}{\sum\limits_{i=1}^{n-k+1} V_i}$
- $m_1 = 0.500 = \frac{320+300+280}{800+600+400}$
- $m_2 = 0.300$
- $m_3 = 0.100$

◇ Calculate $E[U_i^{BC}]$:

- $E\left[U_i^{BC}\right] = V_i \cdot \sum\limits_{k=1}^{n} m_k$
- $E\left[U_1^{BC}\right] = 800(0.500 + 0.300 + 0.100) = 720$
- $E\left[U_2^{BC}\right] = 600(0.500 + 0.300 + 0.100) = 540$
- $E\left[U_3^{BC}\right] = 400(0.500 + 0.300 + 0.100) = 360$

◇ Calculate p_i and q_i:

- $p_i = \frac{\sum\limits_{k=1}^{n-i+1} m_k}{\sum\limits_{k=1}^{n} m_k}$
- $p_1 = \frac{0.500+0.300+0.100}{0.500+0.300+0.100} = 1.000$ and $q_1 = 1 - p_1 = 0.000$
- $p_2 = \frac{0.500+0.300}{0.500+0.300+0.100} = 0.889$ and $q_2 = 1 - p_2 = 0.111$
- $p_3 = \frac{0.500}{0.500+0.300+0.100} = 0.556$ and $q_3 = 1 - p_3 = 0.444$

◇ Calculate R_i^{ind}:

- $R_i^{ind} = \frac{q_i}{p_i} \cdot C_{i,n-i+1}$ and $U_i^{ind} = R_i^{ind} + C_{i,n-i+1}$
- $R_1^{ind} = \frac{0}{1} \cdot (320 + 220 + 80) = 0$
- $R_2^{ind} = \frac{0.111}{0.889} \cdot (300 + 200) = 62.430$
- $R_3^{ind} = \frac{0.444}{0.556} \cdot 280 = 223.597$

◇ Calculate R_i^{coll}:

- $R_i^{coll} = q_i \cdot U_i^{BC}$ and $U_i^{coll} = R_i^{coll} + C_{i,n-i+1}$
- $R_1^{coll} = 0(720) = 0$
- $R_2^{coll} = 0.111(540) = 59.94$

- $R_3^{coll} = 0.444(360) = 159.84$

◇ Calculate R_3^{GB}:

- $R_i^{WN} = Z_i^{WN} \cdot R_i^{ind} + (1 - Z_i^{WN}) \cdot R_i^{coll}$, where $Z_i^{WN} = \sum_{k=1}^{n-i+1} m_k$
- $R_1^{WN} = 0$ since $R_1^{ind} = R_1^{coll} = 0$
- $R_2^{WN} = (0.500 + 0.300)(62.430) + (1 - 0.500 - 0.300)(59.94) = 61.932$
- $R_3^{WN} = (0.500)(223.597) + (1 - 0.500)(159.84) = 191.719$

◇ The total Neuhaus loss ratio claims reserves is $61.932 + 191.719 =$ $\boxed{\$253{,}651}$

Solution to part b:

◇ The premium volume is shrinking over time. This may indicate a change in mix of business. Since the Neuhaus method assumes a constant ELR for all accident years, a change in mix of business may violate the constant ELR assumption

2018 #3

Given the following information as of December 31, 2017:

		Cumulative Paid Loss ($000) as of (months)			
AY	Earned Premium ($000)	12	24	36	48
2014	8,000	2,500	3,335	3,942	4,021
2015	8,320	2,100	2,705	3,335	
2016	8,650	3,000	4,113		
2017	9,000	3,500			

- ◇ Assume there is no further development after 48 months
- ◇ $t_i = \sqrt{p_i}$
- ◇ $E[\alpha_2^2(U_2)] = 2,000$

Calculate the mean squared error for both the individual loss ratio method and the collective loss ratio method, and determine which is preferable for estimating R_{2015}.

◇ Create the triangle of incremental losses:

		Incremental Paid Loss ($000) as of (months)			
AY	Earned Premium ($000)	12	24	36	48
2014	8,000	2,500	835	607	79
2015	8,320	2,100	605	630	
2016	8,650	3,000	1,113		
2017	9,000	3,500			

◇ Calculate the m_k's:

- $m_k = \frac{E\left[\sum_{i=1}^{n-k+1} S_{ik}\right]}{\sum_{i=1}^{n-k+1} V_i}$
- $m_1 = 0.327 = \frac{2500+2100+3000+3500}{8000+8320+8650+9000}$
- $m_2 = 0.102$
- $m_3 = 0.076$
- $m_4 = 0.010$

◇ Calculate p_{2015} and q_{2015}:

- $p_i = \frac{\sum_{k=1}^{n-i+1} m_k}{\sum_{k=1}^{n} m_k}$
- $p_{2015} = \frac{0.327+0.102+0.076}{0.327+0.102+0.076+0.010} = 0.981$
- Thus, $q_{2015} = 1 - 0.981 = 0.019$

◇ The MSE for any credible reserve is $\text{mse}(R_i^c) = E[\alpha_i^2(U_i)] \cdot \left[\frac{Z_i^2}{p_i} + \frac{1}{q_i} + \frac{(1-Z_i)^2}{t_i}\right] \cdot q_i^2$

◇ Thus, the MSE for the individual loss ratio method (Z = 1) is $\text{mse}(R_i^c) = 2000 \cdot \left[\frac{1^2}{0.981} + \frac{1}{0.019} + \frac{(1-1)^2}{\sqrt{0.981}}\right] \cdot 0.019^2 = 38.736$

◇ Thus, the MSE for the collective loss ratio method (Z = 0) is $\text{mse}(R_i^c) = 2000 \cdot \left[\frac{0^2}{0.981} + \frac{1}{0.019} + \frac{(1-0)^2}{\sqrt{0.981}}\right] \cdot 0.019^2 = 38.729$

◇ Since the MSE for the collective method is slightly smaller, it is the preferred method

2017 #1

Given the following information as of December 31, 2016:

		Cumulative Reported Loss ($)		
Accident Year	Earned Premium	12 Months	24 Months	36 Months
2014	1,100,000	450,000	585,000	614,250
2015	1,210,000	600,000	840,000	
2016	1,331,000	850,000		

- ◇ Assume no further development after 36 months

Calculate the ultimate losses for each accident year using each of the following methods:

- ◇ Collective loss ratio
- ◇ Individual loss ratio
- ◇ Benktander loss ratio
- ◇ Optimal credible loss ratio

Solution:

◇ To use Hürlimann's method, we need to calculate incremental losses:

	Incremental Loss Payments ($)		
Accident Year	12 Months	24 Months	36 Months
2014	450,000	135,000	29,250
2015	600,000	240,000	
2016	850,000		

◇ Calculate the m_k's:

- $m_k = \frac{E\left[\sum_{i=1}^{n-k+1} S_{ik}\right]}{\sum_{i=1}^{n-k+1} V_i}$
- $m_1 = 0.522 = \frac{450+600+850}{1100+1210+1331}$
- $m_2 = 0.162$
- $m_3 = 0.027$

◇ Calculate $E[U_i^{BC}]$:

- $E\left[U_i^{BC}\right] = V_i \cdot \sum_{k=1}^{n} m_k$
- $E\left[U_1^{BC}\right] = 1100000(0.522 + 0.162 + 0.027) = 782100$
- $E\left[U_2^{BC}\right] = 1210000(0.522 + 0.162 + 0.027) = 860310$
- $E\left[U_3^{BC}\right] = 1331000(0.522 + 0.162 + 0.027) = 946341$

◇ Calculate the p_i's and q_i's:

- $p_i = \frac{\sum_{k=1}^{n-i+1} m_k}{\sum_{k=1}^{n} m_k}$
- $p_1 = \frac{0.522+0.162+0.027}{0.522+0.162+0.027} = 1.000$ and $q_1 = 1 - p_1 = 0.000$
- $p_2 = \frac{0.522+0.162}{0.522+0.162+0.027} = 0.962$ and $q_2 = 1 - p_2 = 0.038$
- $p_3 = \frac{0.522}{0.522+0.162+0.027} = 0.734$ and $q_3 = 1 - p_3 = 0.266$

◇ Calculate the U_i^{ind}'s:

- $R_i^{ind} = \frac{q_i}{p_i} \cdot C_{i,n-i+1}$ and $U_i^{ind} = R_i^{ind} + C_{i,n-i+1}$
- $R_1^{ind} = \frac{0}{1} \cdot 614250 = 0$. Thus, $U_1^{ind} = 0 + 614250 = \boxed{\$614{,}250}$

- $R_2^{ind} = \frac{0.038}{0.962} \cdot 840000 = 33180.873$. Thus, $U_2^{ind} = 33180.873 + 840000 = \boxed{\$873{,}180.87}$
- $R_3^{ind} = \frac{0.266}{0.734} \cdot 850000 = 308038.147$. Thus, $U_3^{ind} = 308038.147 + 850000 = \boxed{\$1{,}158{,}038.15}$

◇ Calculate the U_i^{coll}'s:

- $R_i^{coll} = q_i \cdot U_i^{BC}$ and $U_i^{coll} = R_i^{coll} + C_{i,n-i+1}$
- $R_1^{coll} = 0(782100) = 0$. Thus, $U_1^{coll} = 0 + 614250 = \boxed{\$614{,}250}$
- $R_2^{coll} = 0.038(860310) = 32691.780$. Thus, $U_2^{coll} = 32691.780 + 840000 = \boxed{\$872{,}691.78}$
- $R_3^{coll} = 0.266(946341) = 251726.706$. Thus, $U_3^{coll} = 251726.706 + 850000 = \boxed{\$1{,}101{,}726.71}$

◇ Calculate the U_i^{GB}'s:

- $U_i^{GB} = Z_i^{GB} \cdot U_i^{ind} + (1 - Z_i^{GB}) \cdot U_i^{coll}$, where $Z_i^{GB} = p_i$
- $U_1^{GB} = p_1 \cdot U_1^{ind} + (1 - p_1) \cdot U_1^{coll} = 1.000(614250) + (1 - 1)(614250) = \boxed{\$614{,}250}$
- $U_2^{GB} = p_2 \cdot U_2^{ind} + (1 - p_2) \cdot U_2^{coll} = 0.962(873180.87) + (1 - 0.962)(872691.78) = \boxed{\$873{,}162.28}$
- $U_3^{GB} = p_3 \cdot U_3^{ind} + (1 - p_3) \cdot U_3^{coll} = 0.734(1158038.15) + (1 - 0.734)(1101726.71) = \boxed{\$1{,}143{,}059.31}$

◇ Calculate the U_i^{opt}'s:

- $U_i^{opt} = Z_i^* \cdot U_i^{ind} + (1 - Z_i^*) \cdot U_i^{coll}$, where $Z_i^* = \frac{p_i}{p_i + \sqrt{p_i}}$
- $U_1^{opt} = \left(\frac{1}{1+\sqrt{1}}\right) \cdot U_1^{ind} + \left(1 - \frac{1}{1+\sqrt{1}}\right) \cdot U_1^{coll} = 0.5(614250) + (1 - 0.5)(614250) = \boxed{\$614{,}250}$
- $U_2^{opt} = \left(\frac{0.962}{0.962+\sqrt{0.962}}\right) \cdot U_2^{ind} + \left(1 - \frac{0.962}{0.962+\sqrt{0.962}}\right) \cdot U_2^{coll} = 0.495(873180.87) + (1 - 0.495)(872691.78) = \boxed{\$872{,}933.88}$
- $U_3^{opt} = \left(\frac{0.734}{0.734+\sqrt{0.734}}\right) \cdot U_3^{ind} + \left(1 - \frac{0.734}{0.734+\sqrt{0.734}}\right) \cdot U_3^{coll} = 0.461(1158038.15) + (1 - 0.461)(1101726.71) = \boxed{\$1{,}127{,}686.28}$

2016 #1

Given the following information:

	Cumulative Loss Payments ($)		
Accident Year	12 Months	24 Months	36 Months
2013	1,500	2,700	3,450
2014	1,600	2,740	
2015	1,700		

- ◇ Exposures and premium are constant across all accident years
- ◇ There is no development beyond 36 months

a) Calculate the total reserve indication as of December 31, 2015 using loss-ratio based payout factors and the Benktander method.

b) Calculate the fifth-iteration Benktander reserve indication for accident year 2015.

c) Assuming $Var(U_i) = Var(U_i^{BC})$, use Hürlimann's method for optimal credibility and minimum variance to calculate the reserve indication for accident year 2015.

Solution to part a:

◇ To use Hürlimann's method, we need to calculate incremental losses:

	Incremental Loss Payments ($)		
Accident Year	12 Months	24 Months	36 Months
2013	1,500	1,200	750
2014	1,600	1,140	
2015	1,700		

◇ Calculate the m_k's (since we are not given a premium, I assumed it was 5000):

- $m_k = \frac{E\left[\sum_{i=1}^{n-k+1} S_{ik}\right]}{\sum_{i=1}^{n-k+1} V_i}$
- $m_1 = 0.320 = \frac{1500+1600+1700}{5000+5000+5000}$
- $m_2 = 0.234$
- $m_3 = 0.150$

◇ Calculate $E[U_i^{BC}]$:

- $E\left[U_i^{BC}\right] = V_i \cdot \sum_{k=1}^{n} m_k$
- $E\left[U_1^{BC}\right] = E\left[U_2^{BC}\right] = E\left[U_3^{BC}\right] = 5000(0.320 + 0.234 + 0.150) = 3520$

◇ Calculate the p_i's and q_i's:

- $p_i = \frac{\sum_{k=1}^{n-i+1} m_k}{\sum_{k=1}^{n} m_k}$
- $p_1 = \frac{0.320+0.234+0.150}{0.320+0.234+0.150} = 1.000$ and $q_1 = 1 - p_1 = 0.000$
- $p_2 = \frac{0.320+0.234}{0.320+0.234+0.150} = 0.787$ and $q_2 = 1 - p_2 = 0.213$
- $p_3 = \frac{0.320}{0.320+0.234+0.150} = 0.455$ and $q_3 = 1 - p_3 = 0.545$

◇ Calculate the R_i^{ind}'s:

- $R_i^{ind} = \frac{q_i}{p_i} \cdot C_{i,n-i+1}$
- $R_1^{ind} = \frac{0}{1} \cdot 3450 = 0$
- $R_2^{ind} = \frac{0.213}{0.787} \cdot 2740 = 741.576$
- $R_3^{ind} = \frac{0.545}{0.455} \cdot 1700 = 2036.264$

◇ Calculate the R_i^{coll}'s:

- $R_i^{coll} = q_i \cdot U_i^{BC}$
- $R_1^{coll} = 0(3520) = 0$
- $R_2^{coll} = 0.213(3520) = 749.760$
- $R_3^{coll} = 0.545(3520) = 1918.400$

◇ Calculate the R_i^{GB}'s:

- $R_i^{GB} = Z_i^{GB} \cdot R_i^{ind} + (1 - Z_i^{GB}) \cdot R_i^{coll}$, where $Z_i^{GB} = p_i$
- $R_1^{GB} = p_1 \cdot R_1^{ind} + (1 - p_1) \cdot R_1^{coll} = 1.000(0) + (1 - 1)(0) = 0$
- $R_2^{GB} = p_2 \cdot R_2^{ind} + (1 - p_2) \cdot R_2^{coll} = 0.787(741.576) + (1 - 0.787)(749.760) = 743.319$
- $R_3^{GB} = p_3 \cdot R_3^{ind} + (1 - p_3) \cdot R_3^{coll} = 0.455(2036.264) + (1 - 0.455)(1918.400) = 1972.028$
- Total reserve $= 0 + 743.319 + 1972.028 = \boxed{\$2{,}715.35}$

Solution to part b:

◇ The Benktander reserve is the **second** iteration of Hürlimann's method

◇ To calculate the third iteration reserve for AY 2015, we apply q_3 to the Benktander AY 2015 ultimate loss. Thus, the third iteration is reserve is $0.545(1700 + 1972.028) = 2001.255$

◇ To calculate the fourth iteration reserve for AY 2015, we apply q_3 to the third iteration AY 2015 ultimate loss. Thus, the fourth iteration reserve is $0.545(1700 + 2001.255) = 2017.184$

◇ To calculate the fifth iteration reserve for AY 2015, we apply q_3 to the fourth iteration AY 2015 ultimate loss. Thus, the fifth iteration reserve is $0.545(1700 + 2017.184) = \boxed{\$2{,}025.87}$

Solution to part c:

◇ Calculate Z_i^*:

- $Z_i^* = \frac{p_i}{p_i + t_i}$
- $Z_3^* = \frac{p_3}{p_3 + t_3} = \frac{0.455}{0.455 + \sqrt{0.455}} = 0.403$

◇ Calculate the optimal reserves (call these R_3^{opt}):

- $R_i^{opt} = Z_i^* \cdot R_i^{ind} + (1 - Z_i^*) \cdot R_i^{coll}$
- $R_3^{opt} = Z_3^* \cdot R_3^{ind} + (1 - Z_3^*) \cdot R_3^{coll} = 0.403(2036.264) + (1 - 0.403)(1918.400) = \boxed{\$1{,}965.90}$