

Exam 7 Cookbook

2022 Sitting

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Introduction

When I took Exam 7 in 2013, I wrote the original version of the Exam 7 Cookbook to help me systematically solve calculation problems on exam day. I found this approach to be so useful to help me pass Exam 7 that I created similar “Cookbooks” for myself for Exams 8 and 9, which I also passed on the first sittings to get my FCAS. I updated and expanded the original Exam 7 Cookbook with the goal of helping you pass Exam 7.

I first released the Exam 7 Cookbook for the Spring 2016 sitting. It’s been regularly updated for each sitting since then. My goal is to help you prepare for the exam with less frustration so that you have your best exam sitting yet!

The Structure

Using the syllabus, I identified all of the calculation-based “problem-types” that I believe are reasonably testable. The table of contents lists the problem-types by paper. By exam day, you should know how to solve each one.

Inside, you’ll find a separate section for each testable problem-type. Each section has the following structure with the goal of preparing you to be able to confidently answer a problem on exam day without wasting time trying to “think through” a problem-solving approach before writing the solution.

Original Practice Problem

For each section, there is an original, exam-style problem that demonstrates the problem-type. I wrote these based off of the syllabus papers, focusing on creating high-quality practice problems that are commensurate with the difficulty-level and style that you can expect to see on exam day.

Solution Recipe

The solution recipe solves the practice problem from start to finish and shows the step-by-step approach you should take to answer a similar problem. For each step, you’ll see:

- The description for what to do in the step
- The Formula(s) necessary for the step
- The Formula(s) translated from symbolic notation to plain-English
- Calculations for the step to solve the example problem

Discussion

Each section includes discussion to add clarity and improve your understanding of the problem-solving approach as well as the concepts underlying the problem.

For many problem-types, I point out potential “twists” that could show up on the exam that would make an exam problem more difficult. Since you’ve taken actuarial exams up to this point, you know that straightforward exam problems are more the exception than the rule!

Source

For each problem-type, I include the pages in the syllabus so you can cross-reference for more information.

More Practice

Here, you’ll see references to past CAS problems and Rising Fellow (RF) practice problems from the Practice Problem Bank. You’ll find this helpful especially closer to the exam if there are particular types of problems that you are struggling with. This section includes references to problems from the 2011-present exams, which are the past exams in the current syllabus structure of Exam 7.

Notation and Formulas

One of the big challenges with Exam 7 is that almost every author uses their own unique set of notation for losses, LDFs, premiums, etc. The changing notation between papers makes preparing for the exam more challenging especially for the following reasons:

- It’s harder to see the big picture and draw connections between the main themes in Exam 7 across multiple papers, which is important to be prepared for higher-level Blooms Taxonomy questions.
- It’s more difficult to have an intuitive understanding of the different methods and how they work, which is critical to be able to solve problems on exam day without wasting time trying to think through an approach.

To help you avoid getting lost in the notation I also show a plain-English version of the formulas in the solution recipe steps. If you’re like me and get lost in the symbolic notation in the syllabus papers, you’ll find that this feature will save you a lot of frustration.

Below are two examples of what the formulas look like in the solution recipe steps:

Mack (2000) – Benktander Method:

$$\boxed{U_{GB} = C_k + q_k U_{BF}} \qquad U_{GB} = Loss + (1 - \%Paid) \times Ult_{BF}$$

Verrall – Incorporating Expert Opinion in the Chain Ladder Method

$$\boxed{E[C_{i,j}] = (\lambda_{i,j} - 1) \cdot D_{i,j-1}} \qquad E[IncLoss_{AY,k}] = (LDF_{AY,k} - 1) \cdot Loss_{AY,k-1}$$

$$\boxed{Var(C_{i,j}) = \varphi \cdot \lambda_{i,j} \cdot (\lambda_{i,j} - 1) \cdot D_{i,j-1}} \qquad Var(IncLoss_{AY,k}) = dispersion \cdot LDF_{AY,k} \cdot (LDF_{AY,k} - 1) \cdot Loss_{AY,k-1}$$

Note that Mack (2000) uses C to indicate cumulative losses and Verrall uses C to indicate incremental losses. In the plain-English version of the formula I make the distinction between incremental and cumulative losses as well as spell out what the symbols in the formulas *really* mean.

I firmly believe you should learn and memorize the formulas in the way that you'll best be able to remember and apply to an exam-day problem. The exam graders want to see that you understand how to apply the different methods, not whether you memorized the specific, symbolic notation from a paper written 20 years ago.

I see no evidence that you would be marked off for writing a formula on the exam with $LDF_{AY,k}$ instead of $\lambda_{i,j}$. In fact, if you look at problem 8 from the 2012 exam, you'll see just that in the sample solutions. Sample solution 1 uses clearer $LDF_{AY,k}$ notation while sample solution 2 uses the Verrall notation, $\lambda_{i,j}$.

How to Best Use the Exam 7 Cookbook

Below is a suggested guide for how you can incorporate the Exam 7 Cookbook in your own study schedule along with the syllabus material and a typical study manual. This is the general approach that I used when I took my fellowship exams.

For each of those exams I had a main study manual as well as the Exam Cookbook, which I built out while I studied for the exam (but you don't need to waste time doing that part!)

First pass through the syllabus

While you're reading a particular paper in the syllabus and your main study manual to learn the material, use the Exam 7 Cookbook to clearly identify what problem-types you need to know from the paper. Study the steps in the solution recipe to learn how to solve the problem-types.

Second pass through the syllabus

Review the steps for the problem-types and make sure you have an intuitive understanding of how to solve the problem. If you also purchased the Exam 7 Problem Pack, start working the practice problems from it as well as past CAS problems.

The first level of understanding is to be able to follow the recipe and nod your head as you read the steps. The next level of understanding is to be able to apply the steps to solve a problem with a blank answer sheet in front of you. At this point, your goal is to begin to build that level of understanding.

Review and Practice Problems (around 6 weeks to 2 weeks before the exam)

At this point you should have a good understanding of the syllabus and how to use the recipe steps to systematically solve the different calculation problems. During this period, you should be doing problems across the whole syllabus and targeting problem-types that you are finding particularly challenging.

By the end of this phase, you might not have all the formulas memorized, but you should know all the steps and how to apply them to solve problems without needing to think too much before starting the solution.

You also should continue building your understanding of the concepts and preparing for essay and Blooms Taxonomy questions. I found it helpful to create flashcards from the papers as well as to re-read sections of the syllabus papers that appear to be likely sources of essay problems.

Final Weeks

In the final weeks, focus on taking practice exams to see problems from the entire syllabus. When taking practice exams, work on your exam strategy to make sure you're able to finish the exam and maximize your points.

Prepare for essay problems in the final weeks by using flashcards to make sure that you know all the details necessary.

Prepare for calculation problems by reviewing the recipes in the Exam 7 Cookbook in a similar fashion to how you use flashcards for essay problems. Using this approach on my fellowship exams, I was able to rapidly review the steps and formulas for how to solve each problem-type that might show up on the exam. This was a huge benefit and gave me a lot of confidence going into the exam.

Exam Day

I used the original Exam Cookbooks together with a traditional study manual using the approach above to take my fellowship exams. On exam day, for almost every calculation problem I was able to start writing the solution without wasting time trying to think through how to solve the problem. I had an intuitive understanding of how to solve each of the problems following the step-by-step recipes.

If you follow this approach, you should be able to develop a similar level of understanding and confidence going into the exam room.

Excel Version for Computer-Based Testing Preparation

For each recipe, there is an accompanying Excel version. Make sure to review those so that you know how to solve problems in the Excel format.

Errata

I always hated seeing errors in study manuals when I studied for exams, so I make every effort to ensure the study materials are accurate. Nevertheless, there may still be some errors in the final version, so I keep an updated errata. Please make sure to check it regularly for any fixes. The link is below:

<https://risingfellow.com/errata>

Getting Unstuck

The syllabus material on the exam is challenging. If you get stuck on a section in the cookbook, I highly recommend first cross-referencing the cookbook section with the syllabus paper. Often that will answer your question quickly and will help you better see the connections between different exam concepts.

If you still are stuck on a section or if you see an error, you can send an email to exam7@RisingFellow.com. I will do my best to get back to you as soon as I'm able to.

Least Squares Method

Brosius

Problem

Given the following information:

Accident Year	Cumulative Paid Losses (\$000)			Earned Premium
	36 Months	48 Months	60 Months	
2009	260	830	1,240	4,120
2010	840	3,540	3,960	5,350
2011	130	1,860	2,840	6,540
2012	2,160	3,240		7,780
2013	3,610			8,010

- The tail factor from 60 months-to-Ultimate is 1.25
- Calculate the estimated unpaid losses for accident year 2013 using the Least Squares method.
 - Calculate the credibility weighting on the link ratio method that the Least Squares method uses for accident years 2012 and 2013.

Solution Recipe

Part a – Least Squares Reserve Estimate

- Convert losses to a cumulative loss ratio triangle and apply the tail factor.

$$\text{Loss Ratio} = \frac{\text{Losses}}{\text{Earned Premium}}$$

$$\text{Loss Ratio}_{2009, 36mo} = \frac{260}{4,120} = 6.31\%$$

$$\text{Ult Loss Ratio}_{2009} = 30.1\% \times 1.25 = 37.6\%$$

Note: This step is only necessary if there's significant exposure change over the accident years.

In this problem, there's significant Earned Prem growth between 2009 and 2013.

AY	36	48	60	Ult L/R
2009	6.31%	20.1%	30.1%	37.6%
2010	15.7%	66.2%	74.0%	92.5%
2011	1.99%	28.4%	43.4%	54.3%
2012	27.8%	41.6%		
2013	45.1%			

- Calculate the least squares a and b parameters for each development period iteratively. Start with the most mature development period. Use undeveloped loss ratios as the 'x' values and ultimate loss ratios as the 'y' values.

$$b = \frac{\overline{xy} - \bar{x} \cdot \bar{y}}{\overline{x^2} - \bar{x}^2}$$

$$a = \bar{y} - b \cdot \bar{x}$$

Starting with the 48mo development period (for accident year 2012):

x (48mo L/R)	y (Ult L/R)		48mo	<u>Excel Function</u>
20.1%	37.6%	\bar{x}	.3825	LINEST(Known Ys, Known Xs)
66.2%	92.5%	\bar{y}	.6148	
28.4%	54.3%	$\overline{x^2}$.1864	Returns b and a
		\overline{xy}	.2808	
$b = \frac{.2808 - .3825 \times .6148}{.1864 - .3825^2} = 1.138$		b	1.138	
$a = .6148 - 1.138 \times .3825 = 0.1795$		a	0.1795	

- 3) Calculate the ultimate loss ratio estimate from the loss ratio to-date for each accident year. Start with the oldest accident year without an ultimate loss ratio.

$$\hat{y} = a + bx$$

$$\widehat{Ult\ Loss\ Ratio} = a + b \times Latest\ Loss\ Ratio$$

$$\hat{y}_{2012} = 0.1795 + 1.138 \times 0.416 = 65.3\%$$

- 4) Repeat steps 2 and 3 iteratively to calculate the ultimate loss ratios for the remaining accident years.

x (36mo L/R)	y (Ult L/R)		36mo	$\hat{y}_{2013} = .4988 + .9709 \times 0.451 = 93.6\%$						
6.31%	37.6%	\bar{x}	.1294							
15.7%	92.5%	\bar{y}	.6244							
1.99%	54.3%	$\overline{x^2}$.0265	<table border="1"> <thead> <tr> <th>AY</th> <th>Ult Loss Ratio</th> </tr> </thead> <tbody> <tr> <td>2012</td> <td>65.3%</td> </tr> <tr> <td>2013</td> <td>93.6%</td> </tr> </tbody> </table>	AY	Ult Loss Ratio	2012	65.3%	2013	93.6%
AY	Ult Loss Ratio									
2012	65.3%									
2013	93.6%									
27.8%	65.3%	\overline{xy}	.0903							
		b	.9709							
		a	.4988							

From Step 3

- 5) Calculate unpaid losses using the estimated ultimate loss ratios.

$$Unpaid\ Loss = Earned\ Prem \times Ult\ Loss\ Ratio - Losses$$

$$Unpaid_{2013} = 8,010 \times 93.6\% - 3,610 = 3,890$$

Part b – Credibility Weighting on Link Ratio Method

- 6) Calculate the LDF-to-ultimate that the link ratio method would use for each year using \bar{y} and \bar{x} .

$$c = LDF = \frac{\bar{y}}{\bar{x}}$$

$$LDF_{2012} = \frac{.6148}{.3825} = 1.607$$

$$LDF_{2013} = \frac{.6244}{.1294} = 4.825$$

7) Calculate the credibility on the link ratio method, Z , for each year.

$$Z = \frac{b}{c}$$

$$Cred = \frac{b}{LDF}$$

$$Z_{2012} = \frac{1.138}{1.607} = 70.8\%$$

$$Z_{2013} = \frac{.9709}{4.825} = 20.1\%$$

Discussion

The least squares method can be used with either actual losses or loss ratios. Brosius recommends using loss ratios if there is significant premium growth. If premium increases normally, then using loss ratios is unnecessary. Using loss ratios puts the accident years on a more equal basis for the least squares method.

The key thing to understand is that steps 2 and 3 are an iterative process. We first use 48mo loss ratios as x and ultimate loss ratios as y in order to get the estimated ultimate loss ratio for AY 2012. When we iterate through with 36mo loss ratios as x , we can use our estimated ultimate loss ratio for AY 2012 (65.3%) as an *additional* data point.

The Least Squares method is a credibility-weighting of the Link Ratio and Budgeted Loss methods with the following formula:

$$\hat{y} = Z \times \frac{x}{d} + (1 - Z) \times E[y]$$

$$\hat{y} = Z \times LDF \times x + (1 - Z) \times \bar{y}$$

AY 2012 Example:

$$\hat{y} = .708 \times 1.607 \times .416 + (1 - .708) \times .6148 = 65.3\% \text{ (same as step 3)}$$

Potential Problems with Parameter Estimation

Sampling error can sometimes result in negative a or b values, which can cause nonsensical loss estimates. If this happens, Brosius recommends the following (see Brosius pg. 4 for more):

The intercept is negative ($a < 0$):

- This causes the estimate of developed losses (\hat{y}) to be negative for small values of x .
- **Solution:** Use the link ratio method instead.

$$LDF = \frac{\bar{y}}{\bar{x}} \quad \hat{y}^{linkratio} = LDF \cdot x$$

The slope is negative ($b < 0$):

- This causes the estimate of y to decrease as x increases
- **Solution:** Use the budgeted loss method instead, ignoring x .

$$\hat{y}^{budget} = \bar{y}$$

Reviewing the Results

As we go from more recent AYs to more mature AYs, we should usually see the following patterns:

- a decreases for more developed years
- c (LDF) decreases for more developed years
- Z increases for more developed years (more weight on the link ratio method)

This is because earlier on (e.g. at 12 months), the actual loss experience (x) is more volatile and less useful in predicting the future loss (y). Therefore, the a parameter is larger and the b parameter is smaller due to less credibility (Z).

As the accident years mature, the actual losses have more credibility and we place greater weight on the link ratio method and less weight on the budgeted loss method. The loss experience receives greater weight and we place less weight on the intercept, a . (See pg. 18 in Brosius for more discussion)

These patterns hold for the calculations above, so this is a good additional check that the results are reasonable.

Source

Brosius – pg. 16-18

More Practice

CAS 2018 – 4	RF Brosius – 1
CAS 2017 – 2	RF Brosius – 3
CAS 2016 – 2	RF Brosius – 5
CAS 2012 – 4	RF Brosius – 8
CAS 2011 – 1	

Bayesian Credibility in a Changing System

Brosius

Problem

Given the following information:

Prior to 2015, the following estimates were used for a small book of personal auto insurance in State X:

- Expected losses were \$35 Million
- Percent of losses reported through 12 months was 70%

A legislative change in the state is estimated to impact expected losses and development patterns going forward.

The actuary estimates the following beginning in 2015:

- Expected losses are estimated to fall by 20%
- Percent of losses reported through 12 months is expected to speed up to 75%
- The actuary selects the standard deviation of losses to be \$5 Million
- The actuary selects the standard deviation of the percent reported at 12 months to be 10%

As of December 31, 2015, accident year 2015 reported losses were \$25 Million.

Calculate the estimated accident year 2015 ultimate losses using Bayesian Credibility.

Solution Recipe

1) Identify the necessary inputs: $E[Y]$, $\sigma(Y)$, $E[\frac{X}{Y}]$, $\sigma(\frac{X}{Y})$

$E[Y]$ → Expected Ultimate Losses

$\sigma(Y)$ → Std Dev of Ultimate Losses

$E[\frac{X}{Y}]$ → Expected %Reported

$\sigma(\frac{X}{Y})$ → Std Dev of %Reported

X → Random variable for losses

Y → Random variable for ultimate losses

$$E[Y] = 35M \times (1 - 0.2) = 28M$$

$$\sigma(Y) = 5M$$

$$E[\frac{X}{Y}] = .75$$

$$\sigma(\frac{X}{Y}) = .10$$

2) Calculate Z , the credibility weight between the Link Ratio and Budgeted Loss (Expected) methods.

$$VHM = (E[\frac{X}{Y}] \cdot \sigma(Y))^2$$

$$EVPV = \sigma(\frac{X}{Y})^2 \cdot (\sigma(Y)^2 + E[Y]^2)$$

$$Z = \frac{VHM}{VHM + EVPV}$$

$$VHM = (.75 \times 5)^2 = 14.06$$

$$EVPV = .10^2 \times [5^2 + 28^2] = 8.09$$

$$Z = \frac{14.06}{14.06 + 8.09} = .635$$

- 3) Calculate ultimate losses as a credibility weighting between the Link Ratio ultimate and the Expected ultimate.

$$Ult = Z \times \frac{X}{E\left[\frac{X}{Y}\right]} + (1 - Z) \times E[Y]$$

$$Ult = Cred \times Ult_{ChainLadder} + (1 - Cred) \times E[Ult]$$

$$Ult = .635 \times \frac{25M}{.75} + (1 - .635) \times 28M = \boxed{31.4M}$$

Discussion

A key assumption of the least squares method is that there aren't systematic shifts in the book of business. With this assumption, historical accident year losses can be used to project ultimate losses for undeveloped accident years.

The Bayesian method is appropriate for a new line of business or when there is a significant change in the book of business and the go-forward experience will be different than historical accident years.

The tricky part with this type of problem will be to properly identify all the inputs necessary to solve the problem. Once you've calculated Z , the credibility weighting, just remember that it's a simple credibility weighting between the chain ladder ultimate and expected ultimate.

Source

Brosius – pg. 14-15

More Practice

CAS 2019 – 1 RF Brosius – 2
CAS 2016 - 2 RF Brosius – 4
CAS 2014 – 1

Caseload Effect

Brosius

Problem

Insurer ABC writes Motorcycle insurance and recently expanded its book of business into a new state. Management expects the written premium to be \$20M in 2016. The reserving actuary makes the following assumptions beginning in 2016:

- The expected loss ratio is 80%
- 70% of losses are expected to be reported by 12 months of development
- The actuary selects the standard deviation of ultimate losses to be \$4M
- The actuary selects the standard deviation of the percent reported at 12 months to be 10%

Due to resource constraints in the claims department, the actuary assumes that if ultimate losses are 25% higher than expected, then the expected percent of losses reported at 12 months will fall to 65%.

As of December 31, 2016, accident year 2016 reported losses were \$13,400,000.

Calculate the estimated accident year 2016 unreported loss reserve using Bayesian Credibility.

Solution Recipe

1) Identify the necessary inputs: $E[Y]$, $\sigma(Y)$, $E[\frac{X}{Y}]$, $\sigma(\frac{X}{Y})$

$E[Y]$ → Expected Ultimate Losses

$\sigma(Y)$ → Std Dev of Ultimate Losses

$E[\frac{X}{Y}]$ → Expected %Reported

$\sigma(\frac{X}{Y})$ → Std Dev of %Reported

X → Random variable for losses

Y → Random variable for ultimate losses

$$E[Y] = 20M \times .8 = 16M$$

$$\sigma(Y) = 4M$$

$$E[\frac{X}{Y}] = .70$$

$$\sigma(\frac{X}{Y}) = .10$$

2) Calculate Z , the credibility weight between the Link Ratio and Budgeted Loss (Expected) methods.

$$VHM = (E[\frac{X}{Y}] \cdot \sigma(Y))^2$$

$$EVPV = \sigma(\frac{X}{Y})^2 \cdot (\sigma(Y)^2 + E[Y]^2)$$

$$Z = \frac{VHM}{VHM + EVPV}$$

$$VHM = (.70 \times 4)^2 = 7.84$$

$$EVPV = .10^2 \times [4^2 + 16^2] = 2.72$$

$$Z = \frac{7.84}{7.84 + 2.72} = .742$$

3) Calculate x_0 and d , the parameters for the caseload effect.

$$\boxed{E[X | Y = y] = dy + x_0} \quad \left| \begin{array}{l} y = 16M: \quad .70 \times 16M = d \times 16M + x_0 \\ y = 1.25 \times 16M = 20M: \quad .65 \times 20M = d \times 20M + x_0 \\ \hline 1.8M = d \times 4M \\ d = 0.45 \quad x_0 = 4M \end{array} \right.$$

4) Calculate the ultimate loss estimate as a credibility weighting between the Link Ratio ultimate and the Expected ultimate using the caseload Bayesian credibility formula.

$$\boxed{\hat{y} = Z \times \frac{x - x_0}{d} + (1 - Z) \times E[Y]}$$

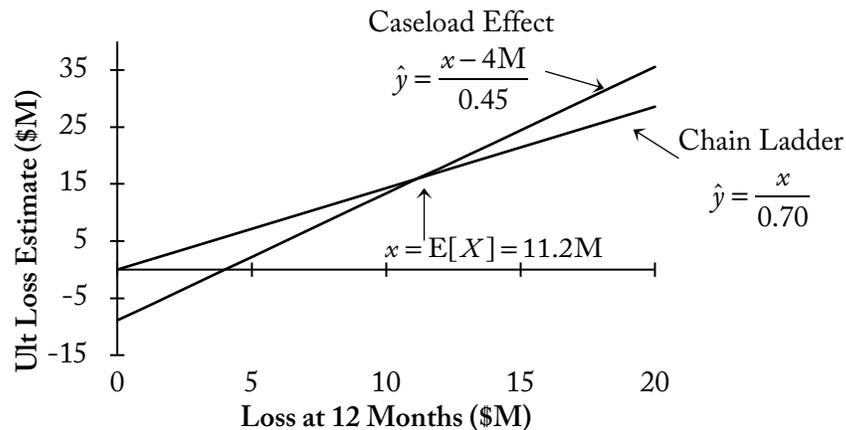
$$\hat{y} = .742 \times \frac{13.4M - 4M}{.45} + (1 - .742) \times 16M = 19.63M$$

$$Loss\ Reserve = 19.63M - 13.4M = \boxed{6.23M}$$

Discussion

This question is similar to the regular Bayesian credibility method, but the chain ladder estimate is modified for the caseload effect. Instead of using a fixed percent reported, the expected percent reported is lower if ultimate losses (Y) are higher. Below is a graphical view of how the caseload effect chain ladder estimate compares to the unmodified chain ladder estimate.

- If $x > E[X]$, the caseload estimate will be higher than the unmodified chain ladder estimate.
- If $x < E[X]$, the caseload estimate will be lower than the unmodified chain ladder estimate.



Source

Brosius – pg. 15-16

More Practice

RF Brosius – 9

RF Brosius – 10

Benktander Method

Mack (2000)

Problem

Given the following information:

Accident Year	Cumulative Paid Losses (\$000)			Earned Premium
	12 Months	24 Months	36 Months	
2012	1,800	2,900	3,350	5,000
2013	2,800	3,600		5,500
2014	2,300			6,000

Selected Loss Development Factors To-Ultimate

12-Ult	24-Ult	36-Ult
1.75	1.25	1.10

- The expected loss ratio is 70%

Calculate the ultimate loss estimate for accident year 2014 using the Benktander method.

Solution Recipe

Method 1 – Using the BF Procedure

- 1) Calculate the ultimate loss estimate for the BF method (1st iteration of the BF procedure).

$$U_{BF} = C_k + q_k U_0$$

$$Ult_{BF} = Loss + (1 - \%Paid) \times Prem \times ELR$$

$$\begin{aligned} U_{BF} &= 2,300 + \left(1 - \frac{1}{1.75}\right) \times 6,000 \times 0.7 \\ &= 4,100 \end{aligned}$$

- 2) Calculate the ultimate loss estimate for the Benktander method (2nd iteration of the BF procedure).

$$U_{GB} = C_k + q_k U_{BF}$$

$$U_{GB} = Loss + (1 - \%Paid) \times Ult_{BF}$$

$$\begin{aligned} U_{GB} &= 2,300 + \left(1 - \frac{1}{1.75}\right) \times 4,100 \\ &= \boxed{4,057} \end{aligned}$$

Method 2 – Credibility-Weighting the Chain Ladder and Expected Loss Ultimates

- 1) Calculate the ultimate loss estimate for the Chain Ladder method.

$$U_{CL} = \frac{C_k}{\hat{p}_k}$$

$$Ult_{CL} = Loss \times CDF$$

$$\begin{aligned} U_{CL} &= 2,300 \times 1.75 \\ &= 4,025 \end{aligned}$$

- 2) Calculate the ultimate loss estimate for the Benktander method. Make sure to use the expected prior loss ultimate (U_0) in the formula.

$$q_k = 1 - \frac{1}{CDF}$$

$$U_{GB} = (1 - q_k^2)U_{CL} + q_k^2 \times U_0$$

$$Ult_{GB} = [1 - \%Unpaid^2] \times Ult_{CL} + \%Unpaid^2 \times Prem \times ELR$$

$$q_k = 1 - \frac{1}{1.75} = .429$$

$$U_{GB} = (1 - .429^2) \times 4,025 + .429^2 \times 6,000 \times .7 = 4,057$$

Method 3 – Credibility-Weighting the Chain Ladder and BF Reserves

- 1) Calculate the reserve loss estimate for the Chain Ladder and BF methods.

$$R_{CL} = \frac{C_k - C_{k-1}}{p_k}$$

$$Resv_{CL} = Loss \times (CDF - 1)$$

$$R_{BF} = q_k U_0$$

$$Resv_{BF} = (1 - \%Paid) \times Prem \times ELR$$

$$R_{CL} = 2,300 \times (1.75 - 1) = 1,725$$

$$R_{BF} = \left(1 - \frac{1}{1.75}\right) \times 6,000 \times 0.7 = 1,800$$

- 2) Calculate the reserve for the Benktander method. Make sure to use the BF reserve in the formula. Add the loss to the reserve estimate to calculate the ultimate loss estimate for the Benktander method.

$$R_{GB} = (1 - q_k)R_{CL} + q_k \times R_{BF}$$

$$Resv_{GB} = [1 - \%Unpaid] \times Resv_{CL} + \%Unpaid \times Resv_{BF}$$

$$R_{GB} = (1 - .429) \times 1,725 + .429 \times 1,800 = 1,757$$

$$U_{GB} = 2,300 + 1,757 = 4,057$$

Discussion

Iterated BF Method

The Benktander method is a second iteration of the BF procedure. This is how the iteration works:

1. Start with an ultimate loss estimate, $U^{(m)}$. For $U^{(0)}$, use the expected loss estimate.
2. Apply the BF procedure to get a new loss reserve estimate:

$$R^{(m)} = q_k U^{(m)}$$

$$Resv^{(m)} = \%Unpaid \times Ult^{(m)}$$

3. Get a new ultimate loss estimate by adding the losses-to-date to the reserve. This is the starting ultimate for the next iteration:

$$U^{(m+1)} = C_k + R^{(m)}$$

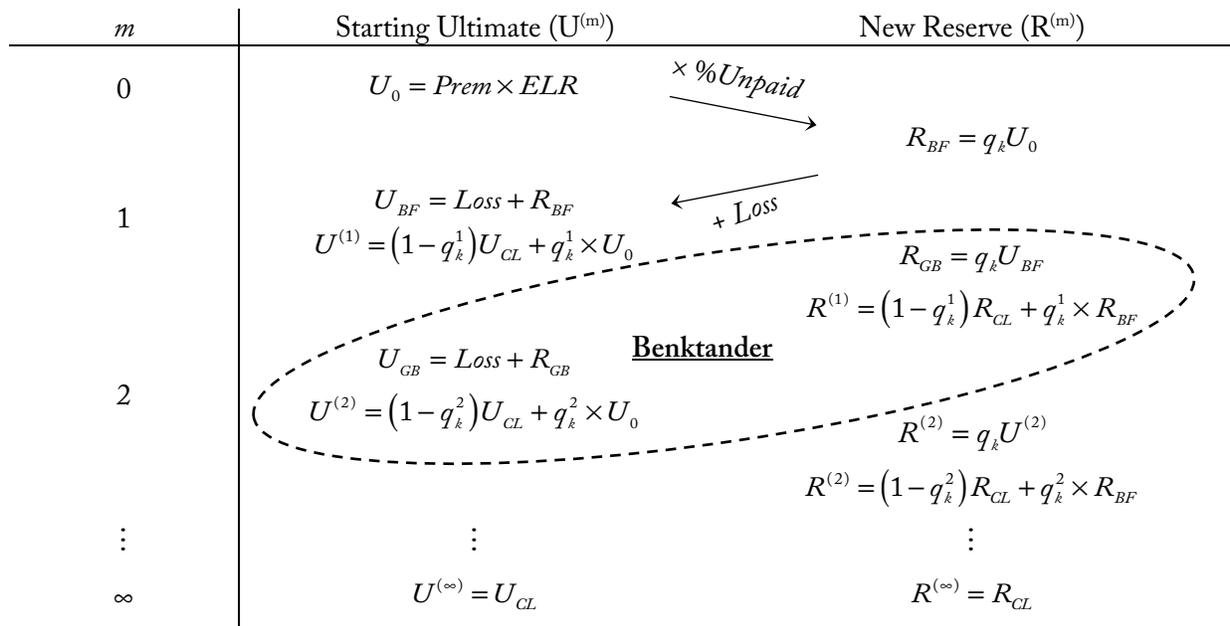
$$Ult^{(m+1)} = Loss_k + Resv^{(m)}$$

The ultimate loss estimate ($U^{(m)}$) can be rearranged as a credibility weighting of the Chain Ladder ultimate (U_{CL}) and expected loss ultimate (U_0). This is Method 2 above.

$$U^{(m)} = (1 - q_k^m)U_{CL} + q_k^m \times U_0$$

Alternatively, the loss reserve estimate ($R^{(m)}$) can be rearranged as a credibility weighting of the Chain Ladder reserve (R_{CL}) and the BF reserve (R_{BF}). This is Method 3 above.

$$R^{(m)} = (1 - q_k^m)R_{CL} + q_k^m \times R_{BF}$$



Note on the iteration number:

Both Hürlimann and Mack (2000) show calculations where the first iteration reserve is labeled R^0 , which is the BF reserve and the second iteration is labeled R^1 , which is the Benktander reserve, so the iteration count is a little confusing.

As the number of iterations increases, the weight on the chain ladder method increases until it converges to the chain ladder method entirely (as $m \rightarrow \infty$).

Source

Mack (2000) – pg. 334-335

More Practice

- CAS 2018 – 5 RF Mack (2000) – 1
- CAS 2016 – 1 RF Mack (2000) – 2
- CAS 2013 – 4 RF Mack (2000) – 3
- CAS 2012 – 1

Credible Loss Ratio Claims Reserves

Hürlimann

Problem

Given the following information:

Accident Year	Incremental Paid Claims (\$000)			Earned Premium
	0-12 Months	12-24 Months	24-36 Months	
2013	180	135	65	450
2014	225	160		475
2015	175			490

Calculate the loss reserve estimate for accident year 2014 for each of the following methods:

- Benktander loss ratio claims reserve
- Neuhaus loss ratio claims reserve
- Optimal credible loss ratio claims reserve

Solution Recipe

- Calculate m_k , the expected incremental loss ratio for each development period.

$$m_k = \frac{\sum IncLoss_{i,k}}{\sum Prem_i}$$

$$m_k = \frac{\sum_{AY=1}^{\# AY's \text{ in dev period } k} IncLoss_{AY,k}}{\sum_{AY=1}^{\# AY's \text{ in dev period } k} Prem_{AY}}$$

$$m_1 = \frac{180 + 225 + 175}{450 + 475 + 490} = .410$$

$$m_2 = \frac{135 + 160}{450 + 475} = .319$$

$$m_3 = \frac{65}{450} = .144$$

- Calculate the expected loss ratio.

$$ELR = \sum m_k$$

$$Expected \ Loss \ Ratio = \sum E[Inc \ Loss \ Ratios]$$

$$ELR = .410 + .319 + .144 = .873$$

- Calculate p_i , the % loss paid for each accident year as of the latest development period.

$$p_i = \frac{\sum m_k}{ELR}$$

$$\% Paid_{AY} = \frac{\sum_{k=1}^{Dev \ Periods \ to-date} m_k}{\sum m_k \text{ for all AY's}}$$

$$p_{2014} = \frac{.410 + .319}{.873} = .835$$

- 4) Calculate R^{ind} and R^{coll} , the reserve estimates for the individual loss ratio and collective loss ratio methods.

$q_i = 1 - \hat{p}_i$	$\%Unpaid_{AY} = 1 - \%Paid_{AY}$	$q_{2014} = 1 - .835$ $= .165$
$R^{ind} = \frac{q_i \cdot Loss_i}{\hat{p}_i}$	$R^{ind} = \frac{\%Unpaid_{AY} \cdot Loss_{AY}}{\%Paid_{AY}}$	$R_{2014}^{ind} = \frac{.165 \times 385}{.835}$ $= 76.3$
$R^{coll} = q_i \cdot Prem \cdot ELR$	$R^{coll} = \%Unpaid_{AY} \cdot Premium_{AY} \cdot ELR$	$R_{2014}^{coll} = .165 \times 475 \times .873$ $= 68.6$

- 5) Calculate credibility weights, Z_i , for each method.

Method	Z_i		
Benktander(Z_i^{GB})	$Z_i^{GB} = \hat{p}_i$	$Z_{2014}^{GB} = .835$	
Neuhaus(Z_i^{WN})	$Z_i^{WN} = \hat{p}_i \cdot ELR$	$Z_{2014}^{WN} = .835 \times .873 = .729$	
Optimal(Z_i^{opt})	$Z_i^{opt} = \frac{\hat{p}_i}{\hat{p}_i + \sqrt{\hat{p}_i}}$	$Z_{2014}^{opt} = \frac{.835}{.835 + \sqrt{.835}} = .477$	Assume: $\text{Var}(U_i) = \text{Var}(U_i^{BC})$

- 6) Calculate the loss reserve estimate as a credibility weighting of R^{ind} and R^{coll} .

$R_i = Z_i \cdot R_i^{ind} + (1 - Z_i) \cdot R_i^{coll}$	$R^{GB} = .835 \times 76.3 + (1 - .835) \times 68.6 = \boxed{75.0}$
	$R^{WN} = .729 \times 76.3 + (1 - .729) \times 68.6 = \boxed{74.2}$
	$R^{opt} = .477 \times 76.3 + (1 - .477) \times 68.6 = \boxed{72.3}$

Discussion

A potential twist to this problem is to use the general version of the optimal credibility formula. The optimal credibility is based on an assumption about the ratio between $\text{Var}(U_i)$ and $\text{Var}(U_i^{BC})$, f_i . If we assume that the variance of the actual ultimate loss, $\text{Var}(U_i)$, is equal to the variance of the burning cost (expected) ultimate loss, $\text{Var}(U_i^{BC})$, then $f_i = 1$, and the optimal credibility weight simplifies to the version above.

If an exam problem gives a different assumption (e.g. $\text{Var}(U_i)$ is 25% greater than $\text{Var}(U_i^{BC})$), then you need to use the general optimal credibility weight formulas to calculate the optimal credibility weight, Z_i^{opt} . See the "Optimal Credibility Weights" recipe for how to do this.

Overall, the Hürlimann method is very similar to the credibility method in Mack (2000), which uses the following as the Benktander loss reserve:

$$R_{GB} = (1 - q_k)R_{CL} + q_k \times R_{BF} \quad \rightarrow \quad R_{GB} = p_k R_{CL} + (1 - p_k)R_{BF}$$

As you can see, the Mack (2000) formula, has the same form as the credibility-weighted formula in Hürlimann: $R_{GB} = p_i R^{ind} + (1 - p_i) \cdot R^{coll}$.

The key difference is that Hürlimann uses expected incremental loss ratios, m_k , to specify the payment pattern (p_i) instead of LDFs. Also, the individual loss ratio reserve is used instead of the chain ladder reserve and the collective loss ratio reserve is used instead of the BF reserve.

Source

See Hürlimann – pg. 82-85 for discussion of the method and pg. 90-91 for the optimal credibility weight. Don't get too sidetracked on all the proofs and notation. Focus on how to apply the method.

More Practice

CAS 2019 – 2	RF Hürlimann – 1
CAS 2019 – 3	RF Hürlimann – 2
CAS 2017 – 1	RF Hürlimann – 3
CAS 2016 – 1	RF Hürlimann – 4
CAS 2015 – 1	
CAS 2013 – 2	

Optimal Credibility Weights

Hürlimann

Problem

Given the following information as of December 31, 2016:

<u>Claims Reserve Estimates</u>		
Accident Year	Individual Loss Ratio Claims Reserve	Collective Loss Ratio Claims Reserve
2014	23,900	24,600
2015	126,700	130,300
2016	566,700	577,800

<u>Loss Ratio Payout Factor (p_i)</u>			
	12 Months	24 Months	36 Months
p_i	76.8%	93.7%	97.4%

- The variance of the ultimate loss is assumed to be 50% greater than the variance of the burning cost ultimate loss estimate.

Calculate the optimal credibility loss ratio claims reserve estimate for accident year 2016.

Solution Recipe

- Calculate the ratio between the variance of the ultimate loss, $\text{Var}(U_i)$, and the variance of the burning cost ultimate loss estimate, $\text{Var}(U_i^{BC})$.

$$f_i = \frac{\text{Var}(U_i)}{\text{Var}(U_i^{BC})}$$

$$f_i = 1.5$$

- Calculate L_i^{opt} based on the variance ratio assumption (step 1) and the % paid loss, p_i .

$$L_i^{opt} = \frac{f_i - 1 + \sqrt{(f_i + 1) \cdot (f_i - 1 + 2p_i)}}{2}$$

$$\begin{aligned} L_{2016}^{opt} &= \frac{1.5 - 1 + \sqrt{(1.5 + 1) \cdot (1.5 - 1 + 2 \times .768)}}{2} \\ &= 1.378 \end{aligned}$$

- Calculate the credibility weights, Z_i^{opt} .

$$Z_i^{opt} = \frac{p_i}{p_i + L_i^{opt}}$$

$$Z_{2016}^{opt} = \frac{.768}{.768 + 1.378} = .358$$

4) Calculate the loss reserve estimate as a credibility weighting of R^{ind} and R^{coll} .

$$R_i = Z_i \cdot R_i^{ind} + (1 - Z_i) \cdot R_i^{coll}$$

$$R_{2016}^{opt} = .358 \times 566,700 + (1 - .358) \times 577,800$$
$$= \boxed{\$573,828}$$

Discussion

This problem is more of a fringe topic and I don't see it as a likely problem.

Notice that the normal optimal credibility weight is a special case of the formulas above, where $f_i = 1$ resulting in $t_i^{opt} = \sqrt{p_i}$. According to Hürlimann, using $f_i = 1$ results in the minimum variance of credible loss reserves among $f_i \geq 1$.

Note on Conflicting Hürlimann Source Papers

The CAS syllabus references the 2009 version of the Hürlimann paper and the Rising Fellow study material is based on this.

The combined PDF of all syllabus papers that's available on the CAS site references the 2009 version but also shows a more recent version of the paper. The newer version shows a slightly different formula for t^{opt} than the original source paper listed in the syllabus.

According to the note in the syllabus, it looks like the newer version of the paper is only for the corrected tables of numbers. Currently, the 2009 version of the Hürlimann paper is what's on the syllabus so I'd prepare based on that. Nevertheless, I'd check back on the CAS site to see if the syllabus reference has changed.

Source

Hürlimann (2009) – pg. 91

More Practice

RF Hürlimann – 3

Variance of Reserves (LDF Method)

Clark

Problem

Given the following information:

Accident Year	Cumulative Paid Losses (\$000)		
	12 Months	24 Months	36 Months
2013	280	400	460
2014	520	605	
2015	460		

- Expected loss emergence is estimated by a Weibull growth curve (in months) with the following parameters:

$$\theta = 8$$

$$\omega = 0.45$$

- Parameter standard deviation of the total estimated reserve is \$120,000
- A truncation point of 10 years is used

Calculate the standard deviation of the total reserve using the LDF method.

Solution Recipe

- Calculate $G(x)$ at each accident year's average age as well as at the truncation point's average age.

Weibull Curve:

$$G(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^\omega}$$

$G(x) \rightarrow \% \text{Rept}$ for average accident time x

Average age (mo) with Accident Years:

$$x_{AY} = \text{Months Development}_{AY} - 6$$

$$x_{2013} = 3 \times 12 - 6 = 30$$

$$G(x) = 1 - e^{-\left(\frac{30}{8}\right)^{0.45}} = .837$$

$$x_{trunc} = 10 \times 12 - 6 = 114$$

$$G(x_{trunc}) = 1 - e^{-\left(\frac{114}{8}\right)^{0.45}} = .963$$

- Calculate the truncated LDF for each accident year.

$$LDF_{trunc} = \frac{G(x_{trunc})}{G(x)}$$

If no truncation:

$$LDF = \frac{1}{G(x)}$$

AY 2013:

$$LDF_{trunc} = \frac{.963}{.837} = 1.15$$

3) Calculate the ultimate loss and loss reserve for each accident year at the truncation point.

$$\boxed{Ult_{AY} = Loss_{AY} \cdot LDF}$$

$$\boxed{Resv_{AY} = Ult_{AY} - Loss_{AY}}$$

Accident Year	Avg Age (x)	Loss	G(x)	LDF _{trunc}	Loss @ 120 mo	Estimated Reserve
2013	30	460	.837	1.15	530	70
2014	18	605	.763	1.26	764	159
2015	6	460	.585	1.65	758	298
Total		1,525			2,051	526

4) [If σ^2 not given] Calculate the expected incremental loss triangle, $\mu_{AY,k}$. Use the true ultimate Ult_{AY} , not the truncated ultimate, when calculating expected incremental losses.

$$\boxed{\mu_{AY,k}^{LDF} = Ult_{AY} [G(x_k) - G(x_{k-1})]}$$

$$E[IncLoss] = Ult_{AY} \cdot \%Incremental Emergence$$

$$\mu_{2014,24} = \frac{605}{.763} \times (.763 - .585) = 141$$

AY	12	24	36	Ult _{AY}
2013	321	98	40	550
2014	463	142		793
2015	460			787
%Incremental Emergence	.585	.179	.074	

5) [If σ^2 not given] Calculate σ^2 , the $\frac{\text{variance}}{\text{mean}}$ ratio for the process variance.

$$\boxed{\sigma^2 = \frac{1}{n-p} \sum \frac{(IncLoss_{AY,k} - \mu_{AY,k})^2}{\mu_{AY,k}}}$$

$$\sigma^2 = \frac{1}{n-p} \sum \frac{\text{triangle}(\text{actual} - \text{expected})^2}{\text{expected}}$$

$n \rightarrow$ # cells in loss triangle

$p \rightarrow$ # parameters

Note: $p^{LDF} = \# \text{AY's} + \# \text{parameters in } G(x)$

Actual Incremental Losses

AY	12	24	36
2012	280	120	60
2013	520	85	
2014	460		

$$p = 3 + 2 = 5 \text{ (3 AY's + 2 } G(x) \text{ parameters)}$$

$$\sigma^2 = \frac{1}{6-5} \left[\frac{(280-321)^2}{321} + \frac{(120-98)^2}{98} + \dots \right]$$

$$= 49.1 \text{ (000's)}$$

6) Calculate the Process Variance.

$$\boxed{ProcessVar = \sigma^2 \cdot Resv}$$

$$ProcessVar = 49.1 \times 526$$

$$= 25,838$$

7) Calculate the standard deviation of the reserve estimate using both the process and parameter variance.

$$\boxed{\text{Std Dev}(\text{Resv}) = \sqrt{\text{Process Var} + \text{Parameter Var}}}$$

$$\begin{aligned} \text{Std Dev}(\text{Resv}) &= \sqrt{25,838 + 120^2} \\ &= 200.6 \end{aligned}$$

Standard deviation of the total reserve is
\$200,600.

Discussion

Be careful with the number of parameters when calculating σ^2 , since the number differs between the LDF and Cape Cod methods. With the LDF method, there is one parameter for the ultimate loss of each accident year in the data and typically two parameters to define the $G(x)$ curve.

A few other things to look out for:

- Make sure to use the *average* age and not the total months of development for x .
- Watch out for cumulative vs. incremental losses in the problem.
- Watch out for switching between thousands and whole dollars in how the inputs are given.
- When calculating σ^2 , make sure to use the untruncated ultimate losses to calculate the expected incremental losses.

If the first development period (t) is less than 12 months, then you need to use the general version of the growth curve:

$$G(t) = \text{expos}(t) \cdot G(x) \quad \text{where} \quad x = \text{AvgAge}(t)$$

	$\text{expos}(t)$	$\text{AvgAge}(t)$
Accident Year	$= \begin{cases} \frac{t}{12} & t \leq 12 \\ 1 & t > 12 \end{cases}$	$= \begin{cases} \frac{t}{2} & t \leq 12 \\ t - 6 & t > 12 \end{cases}$

If the first development period is at 6 months, then the average age of occurrence is 3 months and 50% of the ultimate exposure for the full accident year has been earned: $G(6\text{mo}) = \frac{6}{12} \times G\left(\frac{6}{2}\right) = .5 \times G(3)$

Using Truncation on the Exam

On the exam, I would only use truncation if the problem specifically mentions it, like this one. If it's not explicitly mentioned, I would solve the problem without truncation and save some time. On the 2016 exam for question 3, truncation wasn't mentioned in the problem and the examiner's report said that credit was "neither earned nor lost" for using truncation.

Calculating with whole numbers

The solution here used (000's) throughout. You could also use whole numbers to solve the problem. Below are the key calculations and you can see how it ties with the values above:

$$\sigma^2 = \frac{1}{6-5} \left[\frac{(280,000 - 321,384)^2}{321,384} + \dots \right] = 49,104$$

$$\text{Process Var} = 49,104 \times 526,190 = 2.584 \times 10^{10}$$

$$\text{Std Dev}(\text{Resv}) = \sqrt{2.584 \times 10^{10} + 120,000^2} = \boxed{200,600}$$

Process Variance vs Parameter Variance

- **Process variance** is the variance due to randomness in the insurance process.
- **Parameter variance** is the variance in the estimate of the parameters.

We should see that parameter variance is greater than process variance. This is because most of the uncertainty is due to the inability to estimate the expected reserve (parameter variance) rather than uncertainty due to random events (process variance).

The reason for this is that there are so few data points in a loss triangle to estimate the parameters. Compared to the LDF method, the Cape Cod method (the next recipe) lowers the parameter variance by including the exposure data and using fewer parameters.

Note on the Number of Parameters

LDF Method parameters:

$$p = \#AYs^* + \#\text{parameters in } G(x)$$

* There's one parameter for the Ultimate of each accident year in the LDF method

p represents the number of parameters in the whole model, not just the emergence curve.

For the framework used by Clark, the estimated incremental loss is based on the $G(x)$ curve (2 parameters) and the Ultimate loss for the AY, so these are the parameters in the LDF method. Take a look at pg. 47-48 in Clark for some more discussion about the number of parameters and its importance.

Source

Clark – pg. 59-66

More Practice

CAS 2016 – 3	RF Clark – 1
CAS 2015 – 2	RF Clark – 6
CAS 2014 – 5	RF Clark – 8
CAS 2013 – 3	

Variance of Reserves (Cape Cod Method)

Clark

Problem

Given the following information as of December 31, 2015:

Accident Year	Incremental Paid Losses (\$000)			On-Level Earned Prem (\$000)
	12 Months	24 Months	36 Months	
2013	200	70	40	750
2014	290	80		800
2015	265			810

- Expected loss emergence is estimated by a Loglogistic curve with the following parameters:

$$\theta = 5$$

$$\omega = 0.7$$

- Parameter standard deviation is \$80,000
- A truncation point of 10 years is used to limit loss development

Calculate the coefficient of variation of the total reserve estimate using the Cape Cod method.

Solution Recipe

- Calculate $G(x)$ at each accident year's average age as well as at the truncation point's average age.

Loglogistic Curve:

$$G(x) = \frac{x^\omega}{x^\omega + \theta^\omega}$$

$$x_{2013} = 3 \times 12 - 6 = 30$$

$$x_{trunc} = 12 \times 10 - 6 = 114$$

$$G(x) = \frac{30^{0.7}}{30^{0.7} + 5^{0.7}} = 0.778$$

$$G(x_{trunc}) = 0.899$$

- Calculate the *ELR*, the expected loss ratio.

$$Used\ Prem = G(x) \cdot Prem$$

$$Used\ Prem_{AY} = \%Paid \cdot Premium_{AY}$$

$$ELR = \frac{\sum Loss_{AY}}{\sum Used\ Prem_{AY}}$$

AY	Avg Age(x)	G(x)	Loss-to-date	Prem	Used Prem
2013	30	.778	310	750	584
2014	18	.710	370	800	568
2015	6	.532	265	810	431
			945		1,583

$$ELR = \frac{945}{1,583} = 59.7\%$$

3) Calculate the estimated loss reserve for each accident year with the Cape Cod method.

$$\%Unpaid = G(x_{trunc}) - G(x)$$

$$Expected\ Loss = Prem \cdot ELR$$

$$Reserve = \%Unpaid \cdot Expected\ Loss$$

Note: %Unpaid is limited at the truncation point

$$\%Unpaid_{2013} = .899 - .778 = .121$$

$$Expected\ Loss_{2013} = 750 \cdot 59.7\% = 448$$

$$Reserve_{2013} = .121 \cdot 448 = 54.2$$

AY	%Unpaid	Expected Loss	Reserve
2013	.121	448	54
2014	.189	478	90
2015	.367	484	178
			322

4) [If σ^2 not given] Calculate the expected incremental loss triangle, $\mu_{AY,k}$.

$$\mu_{AY,k}^{CC} = Prem \cdot ELR \cdot [G(x_k) - G(x_{k-1})]$$

$$E[IncLoss] = Expected\ Loss \cdot \%Incremental\ Emergence$$

$$\mu_{2014,24} = 478 \cdot (.710 - .532) = 85.1$$

Expected Incremental Losses

AY	12	24	36	Expected Loss
2013	238	80	30	448
2014	254	85		478
2015	257			484
%Incremental Emergence	.532	.178	.068	

5) [If σ^2 not given] Calculate σ^2 , the $\frac{\text{variance}}{\text{mean}}$ ratio for the process variance.

$$\sigma^2 = \frac{1}{n-p} \sum \frac{(IncLoss_{AY,k} - \mu_{AY,k})^2}{\mu_{AY,k}}$$

$$\sigma^2 = \frac{1}{n-p} \sum \frac{\text{triangle} (actual - expected)^2}{expected}$$

$$p^{CC} = 1 + \# \text{ parameters in } G(x)$$

$$\sigma^2 = \frac{1}{6-3} \left[\frac{(200-238)^2}{238} + \frac{(70-80)^2}{80} + \dots \right]$$

$$= 5.35$$

6) Calculate the process variance.

$$Process\ Var = \sigma^2 \cdot Resv$$

$$Process\ Var = 5.35 \times 322$$

$$= 1,724$$

- 7) Calculate the standard deviation and coefficient of variation of the reserve estimate using both the process and parameter variance.

$$\text{Std Dev}(Resv) = \sqrt{\text{Process Var} + \text{Parameter Var}}$$

$$CV = \frac{\text{Std Dev}(Resv)}{Resv}$$

$$\text{coefficient of variation} = \frac{\text{standard deviation}}{\text{mean}}$$

$$\text{Std Dev}(Resv) = \sqrt{1,724 + 80^2} = 90.1$$

$$CV = \frac{90.1}{322}$$

$$= \boxed{28.0\%}$$

Discussion

Be careful with the number of parameters when calculating σ^2 . With the Cape Cod method, there is one parameter for the ELR and typically two parameters to define the $G(x)$ curve. A different $G(x)$ curve could have a different number of parameters.

The Cape Cod method assumes a constant ELR across all accident years. You can test this assumption by graphing the estimated ultimate loss ratios by accident year, using the formula below:

$$\overline{\text{Ult Loss Ratio}}_{AY} = \frac{\text{Loss}_{AY}}{\text{Prem}_{AY} \times G(x)}$$

$$\overline{\text{Ult Loss Ratio}}_{CC} = \frac{\text{Loss}_{AY}}{\text{Used Prem}_{AY}}$$

These are the estimated ultimate loss ratios by accident year for the problem above:

AY	Loss-to-date	$G(x)$	Prem	Ult L/R
2013	310	.778	750	53.1% ← $= \frac{310}{.778 \times 750}$
2014	370	.710	800	65.1%
2015	<u>265</u>	.532	<u>810</u>	<u>61.5%</u>
	945		2,360	59.7%

If there is a pattern, such as an increasing loss ratio, then the assumption of a constant ELR isn't reasonable. For this problem, a constant ELR seems reasonable.

Source

Clark – pg. 66-69

More Practice

- | | |
|--------------|--------------|
| CAS 2018 – 6 | RF Clark – 2 |
| CAS 2017 – 4 | RF Clark – 7 |
| CAS 2017 – 5 | RF Clark – 9 |
| CAS 2016 – 3 | |
| CAS 2016 – 4 | |
| CAS 2014 – 5 | |
| CAS 2014 – 3 | |
| CAS 2012 – 2 | |
| CAS 2011 – 2 | |

Normalized Residuals

Clark

Problem

Given the following information:

Accident Year	Incremental Reported Losses		
	12 Months	24 Months	36 Months
2013	420	1,220	1,010
2014	360	1,830	
2015	310		

- Process variance/mean ratio (σ^2): 62.76
- Expected loss emergence was estimated by a Weibull growth curve with the following parameters:

$$\theta = 24$$

$$\omega = 1.55$$

- The LDF method was used to estimate unpaid loss

Graph the following residual plots:

- Normalized residuals against the increment age of loss emergence
- Normalized residuals against expected incremental loss
- Normalized residuals against calendar year

Solution Recipe

- Calculate the expected ultimate losses by accident year.

Weibull Curve:

$$G(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^\omega}$$

$$LDF = \frac{1}{G(x)}$$

$$Ult_{AY} = Loss_{AY} \cdot LDF$$

$$G(x_{2014}) = 1 - e^{-\left(\frac{18}{24}\right)^{1.55}} = .473$$

$$LDF_{2014} = \frac{1}{.473} = 2.115$$

$$Ult_{2014} = (360 + 1,830) \times 2.115 = 4,632$$

AY	Avg Age(x)	Loss	G(x)	LDF	Ult
2013	30	2,650	.757	1.322	3,502
2014	18	2,190	.473	2.115	4,632
2015	6	310	.110	9.084	2,816

2) Calculate the expected incremental loss triangle, $\mu_{AY,k}$.

$$\mu_{AY,k}^{LDF} = Ult_{AY} [G(x_k) - G(x_{k-1})]$$

$$E[IncLoss] = Ult_{AY} \cdot \%Incremental\ Emergence$$

AY	12	24	36	Ult_{AY}
2013	386	1,270	994	3,502
2014	510	1,680		4,632
2015	310			2,816
%Incremental Emergence	.110	.363	.284	

3) Calculate the normalized residuals between actual and expected incremental losses.

$$r_{AY,k} = \frac{IncLoss_{AY,k} - \mu_{AY,k}}{\sqrt{\sigma^2 \cdot \mu_{AY,k}}}$$

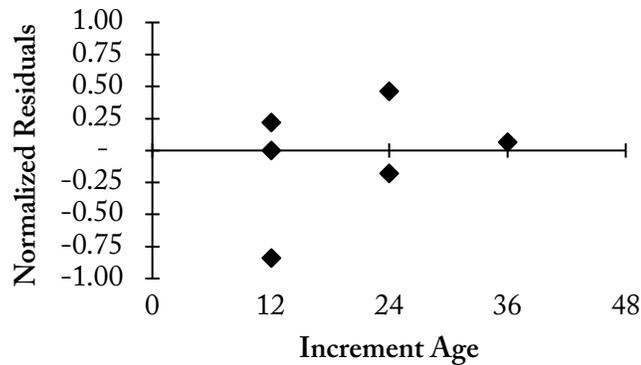
$$Norm.residual = \frac{actual\ incremental - expected\ incremental}{\sqrt{\sigma^2 \cdot expected\ incremental}}$$

$$r_{2014,24} = \frac{1,830 - 1,680}{\sqrt{62.76 \cdot 1,680}} = .462$$

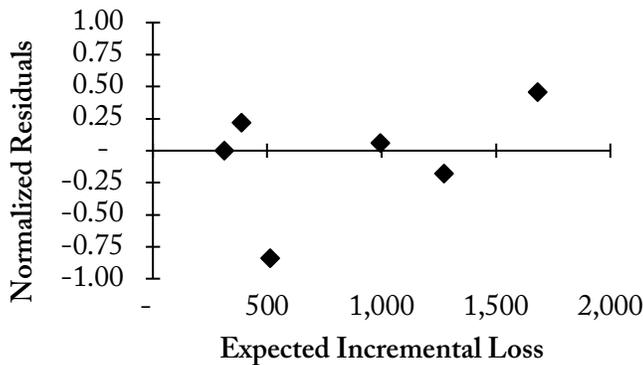
AY	12	24	36
2013	.221	-.179	.064
2014	-.838	.462	
2015	0.000		

4) Graph the relevant residual plots.

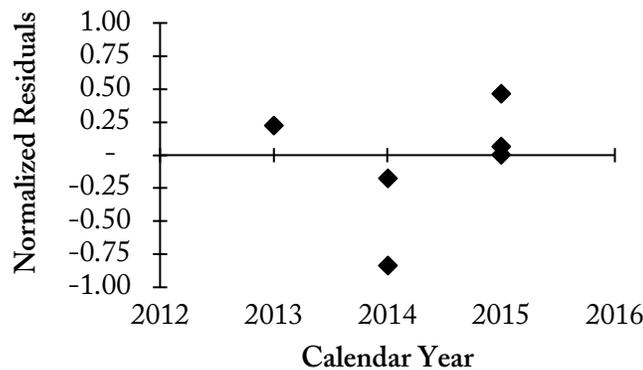
a) Normalized Residuals vs. Increment Age



b) Normalized Residuals vs. Expected Incremental Loss



c) Normalized Residuals vs. Calendar Year



Calendar Year	Normalized Residual
2013	.221
2014	-.179
2014	-.838
2015	.064
2015	.462
2015	0.000

Discussion

Residual graphs are used to test the assumptions in the model. The normalized residuals should be randomly scattered around zero. Look for patterns or trends of positive/negative residuals.

A few things to look for:

- **Normalized Residuals vs. Increment Age:** Use this graph to test how well the loss emergence curve $G(x)$ fits incremental losses at different development periods. See whether the curve over-estimates for some development periods and under-estimates for others.
- **Normalized Residuals vs. Expected Incremental Loss:** Use this graph to check the assumption that the variance/mean scale parameter, σ^2 , is constant. If the residuals are clustered closer to zero at either high or low expected incremental losses, this assumption may not be appropriate.
- **Normalized Residuals vs. Calendar Year:** Use this graph to test for diagonal effects. You might see particularly high or low residuals for a specific calendar year. In the graph above, calendar year 2014 has negative residuals. This might be evidence of calendar year effects that are resulting in lower losses than expected.

Source

Clark – pg. 62-63

More Practice

CAS 2019 – 6 RF Clark – 3
 CAS 2013 – 3

Variance of Prospective Losses

Clark

Problem

Next year's plan premium for an insurer's auto book is \$9,600,000 with an expected loss ratio of 67.2%.

From the reserve analysis, the Cape Cod method is used with LDF curve fitting. Process variance is calculated with a variance/mean ratio of 74,000.

The parameter variance is calculated using the following covariance matrix from the model:

$$\begin{array}{c} ELR \\ \omega \\ \theta \end{array} \begin{array}{c} ELR \\ \omega \\ \theta \end{array} \begin{bmatrix} 0.0028 & -0.0035 & 0.2648 \\ -0.0035 & 0.0093 & -0.4235 \\ 0.2648 & -0.4235 & 31.5681 \end{bmatrix}$$

Calculate the coefficient of variation of losses for the prospective year.

Solution Recipe

1) Calculate the expected losses for the prospective year.

$$E[Loss_{Prosp}] = Prem \cdot ELR$$

$$\begin{aligned} E[Loss_{Prosp}] &= 9,600,000 \times 67.2\% \\ &= 6,451,200 \end{aligned}$$

2) Calculate the process variance.

$$ProcessVar = \sigma^2 \cdot E[Loss_{Prosp}]$$

$$\begin{aligned} ProcessVar &= 74,000 \times 6,451,200 \\ &= 4.774 \times 10^{11} \end{aligned}$$

3) Calculate the parameter variance using the variance of the ELR.

$$ParameterVar = Var(ELR) \cdot Prem^2$$

$$\begin{aligned} ParameterVar &= 0.0028 \times 9,600,000^2 \\ &= 2.580 \times 10^{11} \end{aligned}$$

*See note in the discussion about the derivation of the parameter variance formula.

4) Calculate the standard deviation and coefficient of variation of prospective losses.

$$\text{StdDev}(Loss_{Prosp}) = \sqrt{\text{ProcessVar} + \text{ParameterVar}}$$

$$CV = \frac{\text{StdDev}(Loss_{Prosp})}{Resv}$$

$$\text{coefficient of variation} = \frac{\text{standard deviation}}{\text{mean}}$$

$$\begin{aligned} \text{StdDev}(Loss_{Prosp}) &= \sqrt{4.774 \times 10^{11} + 2.580 \times 10^{11}} \\ &= 857,576 \end{aligned}$$

$$\begin{aligned} CV &= \frac{857,576}{6,451,200} \\ &= \boxed{13.3\%} \end{aligned}$$

Discussion

This problem type is similar to Clark problems about calculating the variance of loss reserves. First, we calculate the prospective expected losses with the expected loss ratio. Then, this is treated similarly to the estimated loss reserve in the other problems.

An exam problem might show a covariance matrix instead of the stand-alone $\text{Var}(\text{ELR})$. Just grab the variance of the ELR from matrix.

Derivation of the Prospective Loss Parameter Variance Formula (Step 3)

We're giving the parameter variance of the ELR, which is the parameter variance of the prospective Expected Loss-to-Premium ratio, but we need the parameter variance of the prospective Expected Loss.

$$\text{Var}(ELR) = E[ELR^2] - E[ELR]^2$$

- We're given $\text{Var}(ELR)$

$$\text{Var}\left(\frac{Loss}{Prem}\right) = E\left[\frac{Loss^2}{Prem^2}\right] - E\left[\frac{Loss}{Prem}\right]^2$$

- $ELR = Loss/Prem$

$$= \frac{E[Loss^2]}{Prem^2} - \frac{E[Loss]^2}{Prem^2}$$

- $Prem$ is a constant

$$\text{Var}\left(\frac{Loss}{Prem}\right) \cdot Prem^2 = E[Loss^2] - E[Loss]^2$$

- Multiply both sides by $Prem^2$

$$\text{Var}\left(\frac{Loss}{Prem}\right) \cdot Prem^2 = \text{Var}(Loss)$$

- $\text{Var}(X) = E[X^2] - E[X]^2$

$$\therefore \text{ParameterVar} = \text{Var}(Loss) = \text{Var}(ELR) \cdot Prem^2$$

- Rearrangement

Source

Clark – pg. 70

More Practice

CAS 2019 – 5

RF Clark – 4

Variance of Calendar Year Development

Clark

Problem

Below is a summary of estimated ultimate losses from the LDF method as of December 31, 2015:

Accident Year	Reported Losses (\$000)	Estimated Ultimate (\$000)
2012	3,100	3,339
2013	3,300	4,143
2014	2,600	4,972
2015	600	4,520

- The expected accident year loss emergence is estimated by a Weibull growth curve with the following parameters:

$$\theta = 22$$

$$\omega = 1.5$$

- Process variance is calculated with a variance/mean ratio of \$30,770
- Parameter standard deviation for 12-month development is \$500,000

Calculate the total standard deviation of estimated loss development over the next 12 months.

Solution Recipe

- Calculate $G(x)$ at each accident year's average age and at the average age after the calendar year development.

Weibull Curve:

$$G(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^\omega}$$

$$x_{2013} = 36 - 6 = 30$$

$$G(x) = 1 - e^{-\left(\frac{30}{22}\right)^{1.5}} = .7966$$

$$G(x+12) = 1 - e^{-\left(\frac{42}{22}\right)^{1.5}} = .9285$$

- Calculate the estimated development for the next calendar period.

LDF Method:

$$\text{Est Development} = \text{Ult} \cdot [G(x+12) - G(x)]$$

$$\text{Est Development} = \text{Est Ult} \cdot \% \text{Incremental Emergence}$$

Cape Cod Method:

$$\text{Est Development} = \text{Prem} \cdot \text{ELR} \cdot [G(x+12) - G(x)]$$

$$\begin{aligned} \text{Est Development}_{2013} &= 4,143 \times [.9285 - .7966] \\ &= 546 \end{aligned}$$

AY	Reported Losses	Avg Age (x)	Avg Age after development ($x + 12$)	$G(x)$	$G(x + 12)$	Ult_{AY}	Estimated Development
2012	3,100	42	54	.9285	.9786	3,339	167
2013	3,300	30	42	.7966	.9285	4,143	546
2014	2,600	18	30	.5229	.7966	4,972	1,361
2015	600	6	18	.1327	.5229	4,520	1,764
							3,838

3) Calculate the process variance for the development period.

$$\text{Process Var} = \sigma^2 \cdot \text{Resv}$$

$$\begin{aligned} \text{Process Var} &= 30,770 \times 3,838,000 \\ &= 1.181 \times 10^{11} \end{aligned}$$

4) Calculate the total standard deviation of the estimated loss development.

$$\text{Std Dev}(\text{Resv}) = \sqrt{\text{Process Var} + \text{Parameter Var}}$$

$$\begin{aligned} \text{Std Dev}(\text{Resv}) &= \sqrt{1.181 \times 10^{11} + 500,000^2} \\ &= \boxed{606,700} \end{aligned}$$

Discussion

The advantage of this type of calculation is that the model can be tested in a relatively short period of time. After one year, the actual 12-month loss development can be compared to the original forecasted range to test whether the actual development falls within the range.

One twist to this type of problem would be to use the Cape Cod method instead of the LDF method.

Source

Clark – pg. 71

More Practice

RF Clark – 5

Finding Best-Fit Parameters with MLE

Clark

Problem

Given the following information as of December 31, 2016:

Accident Year	Incremental Losses		
	12 Months	24 Months	36 Months
2014	3,600	3,230	380
2015	4,200	3,660	
2016	3,620		

Expected loss emergence is estimated by a Loglogistic growth curve (in months). The Loglogistic curve is being fit to the incremental losses using maximum likelihood. You are testing the fit of the following set of parameters:

$$\theta = 10$$

$$\omega = 1.1$$

Calculate the sum of the MLE term for this set of test parameters using the LDF method.

Solution Recipe

- 1) Calculate $G(x)$ at each accident year, using the average age, x .

Loglogistic Curve:

$$G(x) = \frac{x^\omega}{x^\omega + \theta^\omega}$$

$$x_{2015} = 2 \times 12 - 6 = 18$$

$$G(x) = \frac{18^{1.1}}{18^{1.1} + 10^{1.1}} = 0.656$$

AY	$G(x)$
2014	.770
2015	.656
2016	.363

- 2) Calculate the ultimate loss estimate for each accident year.

$$Ult_{AY} = \frac{Loss_{AY}}{G(x)}$$

$$Ult_{2015} = \frac{4,200 + 3,660}{.656} = 11,977$$

AY	Ult
2014	9,363
2015	11,977
2016	9,970

3) Calculate the expected incremental loss triangle, $E[IncLoss_i]$.

$$\mu_{AY,k}^{LDF} = Ult_{AY} \cdot [G(x_k) - G(x_{k-1})]$$

$$E[IncLoss] = Ult_{AY} \cdot \%Incremental\ Emergence$$

$$\mu_{2015,24} = 11,977 \times (.656 - .363) = 3,511$$

AY	12	24	36	Ult_{AY}
2014	3,400	2,745	1,065	9,363
2015	4,349	3,511		11,977
2016	3,620			9,970
%Incremental Emergence	.363	.293	.114	

4) Calculate the triangle of the MLE term.

$$MLE\ term = IncLoss_i \cdot \ln(\mu_i) - \mu_i$$

$$MLE\ term = actual \cdot \ln(expected) - expected$$

$$MLE\ term_{2015,24} = 3,660 \times \ln(3,511) - 3,511 = 26,368$$

AY	12	24	36
2014	25,874	22,829	1,584
2015	30,837	26,368	
2016	26,043		

5) Calculate the loglikelihood term to maximize by summing the MLE term over the triangle for the set of parameters.

$$l = \sum IncLoss_i \cdot \ln(\mu_i) - \mu_i$$

$$l = \sum MLE\ term$$

$$l = 25,874 + \dots + 1,584 = 133,534$$

The best-fitting parameters will maximize l .

Discussion

For the Cape Cod method, use the Cape Cod expected incremental loss formula below:

$$\mu_{AY,k}^{CC} = Prem_{AY} \cdot ELR \cdot [G(x_k) - G(x_{k-1})]$$

$$E[IncLoss] = Prem \cdot ELR \cdot \%Incremental\ Emergence$$

The true loglikelihood term includes σ^2 , but since we assume it's fixed, we can find the best-fitting parameters by maximizing the simpler function, l . One potential twist is that the original data could be in thousands. Make sure to use whole numbers because the MLE term uses a logarithm and you won't get the correct solution using thousands.

Source

Clark – pg. 50-52

More Practice

RF Clark – 10

Residual Test

Mack (1994)

Problem

Given the following information:

Cumulative Reported Losses (\$000)

Accident Year	24 Months	36 Months
2009	10,700	12,400
2010	9,500	12,700
2011	15,000	17,600
2012	20,100	22,300
2013	21,000	24,800
2014	11,800	14,700
2015	11,200	

- Create a plot of cumulative losses at 36 months vs. cumulative losses at 24 and test the Mack assumption that $E[C_{i,k+1} | C_{i,1}, \dots, C_{i,k}] = C_{i,k} \cdot f_k$.
- Plot weighted residuals vs. cumulative losses at 24 months with Mack's methodology to test whether the variance of cumulative losses at 36 months is proportional to cumulative losses at 24 months.

Solution Recipe

Part a – Plot of Cumulative Losses from Adjacent Periods

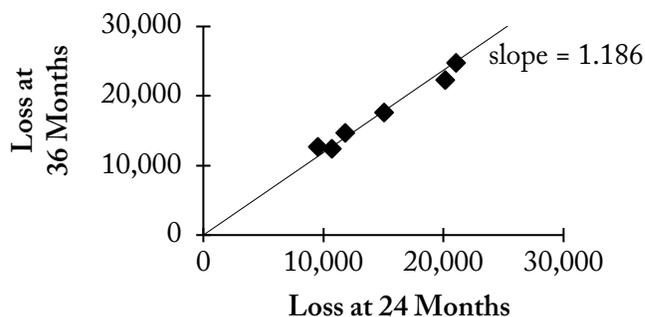
- Calculate the LDF under Mack's variance assumption (volume-weighted LDF).

$$\hat{f}_k = \frac{\sum C_{j,k+1}}{\sum C_{j,k}}$$

$$LDF_k = \frac{\sum Loss_{k+1}}{\sum Loss_k}$$

$$LDF_{24mo} = \frac{12,400 + \dots + 14,700}{10,700 + \dots + 11,800} = 1.186$$

- Plot the cumulative losses at development age k+1 (y-axis) against cumulative losses at development age k (x-axis). Graph a straight line through the origin with a slope of the LDF from step 1. This should fit the data points reasonably well.



The graph shows that cumulative losses at 36 months are linearly proportional to losses at 24 months with a line through the origin, so the assumption is reasonable.

Part b – Weighted Residual Plot

1) Calculate the weighted residuals using the appropriate variance assumption.

$$\varepsilon = \frac{C_{i,k+1} - C_{i,k} \cdot \hat{f}_k}{\sqrt{C_{i,k}}}$$

$$\text{wtd residual} = \frac{\text{Loss}_{k+1} - \text{Loss}_k \cdot \text{LDF}_k}{\sqrt{\text{Variance assumption of Loss}_{k+1}}}$$

Variance Assumptions

$\text{Var}(C_{k+1}) \propto$	LDF calc. Weight	
1	$C_{i,k}^2$	$C_{i,k}^2$ -wtd
$C_{i,k}$	$C_{i,k}$	Vol-weighted
$C_{i,k}^2$	1	Simple Avg

$$\begin{aligned} \varepsilon_{2009} &= \frac{12,400 - 10,700 \times 1.186}{\sqrt{10,700}} \\ &= -2.82 \end{aligned}$$

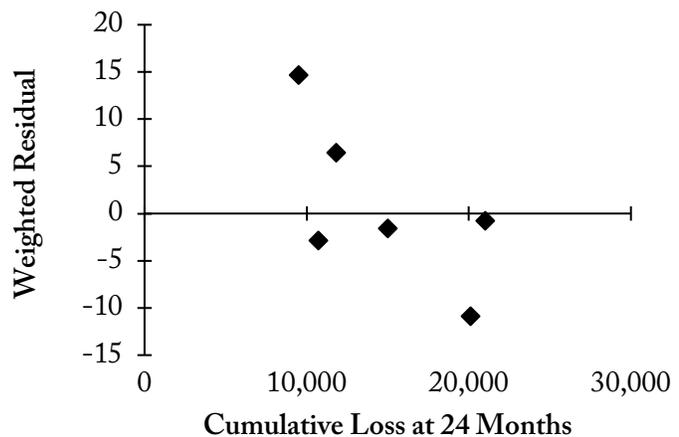
Weighted Residuals

AY	ε_{AY}
2009	-2.82
2010	14.69
2011	-1.57
2012	-10.87
2013	-0.75
2014	6.48

2) Plot the weighted residuals vs. cumulative losses at the prior development period.

Data to plot:

AY	$\text{Loss}_{AY,24}$	ε_{AY}
2009	10,700	-2.82
2010	9,500	14.69
2011	15,000	-1.57
2012	20,100	-10.87
2013	21,000	-0.75
2014	11,800	6.48



3) Interpret the graphs to test the appropriateness of the variance assumption.

The residuals should be random around zero and shouldn't have significant trends or patterns. The graph above appears to show positive residuals for smaller 24-month losses and negative residuals for higher losses, indicating the assumption is not met. But, there are only a few data points so it's hard to tell.

We should create residual plots with the other variance assumptions to see if the residuals show more random behavior.

Discussion

Make sure you have a thorough understanding of the three Mack assumptions so that you can interpret diagnostics like the graphs above and defend whether the assumptions hold or not in an exam question.

The original Mack assumptions use a volume-weighted LDF with the assumption that the variance of the loss at the next development period is proportional to cumulative losses to-date. If a different variance assumption is used, calculate the LDF and the weighted residuals according to the Variance Assumptions table.

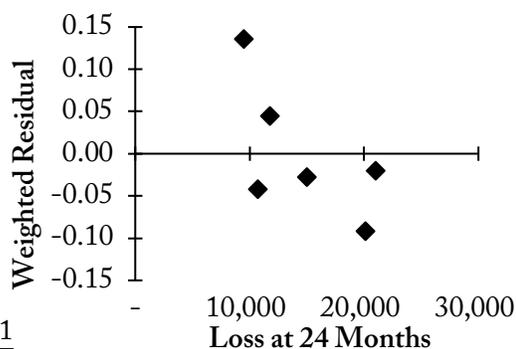
Example of alternative variance assumption

One alternative assumption is that variance is proportional to losses-to-date squared ($\text{Var}(C_{k+1}) \propto C_k^2$). Using this assumption, we calculate the LDF as a simple average of the individual LDFs and then use the modified weighted residual formula using the alternative variance assumption:

$$\epsilon = \frac{C_{i,k+1} - C_{i,k} \cdot \hat{f}_k}{\sqrt{C_{i,k}^2}} = \frac{C_{i,k+1} - C_{i,k} \cdot \hat{f}_k}{C_{i,k}}$$

AY	age-to-age	$Loss_{AY,24}$	ϵ_{AY}
2009	1.159	10,700	-0.04
2010	1.337	9,500	0.14
2011	1.173	15,000	-0.03
2012	1.109	20,100	-0.09
2013	1.181	21,000	-0.02
2014	1.246	11,800	0.04
LDF _{simple avg}	1.201		

$$= \frac{14,700 - 11,800 \times 1.201}{11,800}$$



Comparing the weighted residual graphs using the chain ladder variance assumption and the alternative variance assumption, we don't see much improvement with the alternative assumption.

Source

See pg. 121-124 for discussion about the variance assumptions and weighted residual plots.

See pg. 127-129 for discussion of the paper's numeric example. The plots are shown at the end of the paper.

More Practice

- CAS 2018 – 7
- CAS 2017 – 3
- CAS 2016 – 7
- CAS 2015 – 3
- CAS 2014 – 2
- CAS 2012 – 3
- RF Mack (1994) – 1
- RF Mack (1994) – 7