Contents

Introduction

How To Use This Guide

This guide is intended to **supplement** the syllabus readings. Although I believe it provides a thorough review of the exam material, the readings provide additional context that is invaluable. Please do NOT skip the syllabus readings.

This guide is meant to be used in conjunction with the Rising Fellow (RF) Cookbook, Practice Problem Bank and High-Level Summaries. I suggest you work through RF practice problems as you make your way through the guide. As you near the exam date, the RF High-Level Summaries and Casual Fellow (CF) Flashcards should help you focus in on key topics.

Flashcards

CF Flashcards are provided for each paper. The flashcards are often framed in terms of a question. The intent is for the flashcards to look similar to conceptual questions that might show up on the exam.

Original Practice Problems

There are no original CF practice problems included within this study guide. Instead, the guide was designed to be used in conjunction with the RF practice problem bank.

Past CAS Exam Problems

Past CAS exam problems & solutions from 2016-2019 are included for each paper. Note that these questions are solely owned by the CAS. They are included in the online course for student convenience. All past CAS problems are Excel-based and can be downloaded from the online course.

Website

Outside of the occasional email, all study guide updates (errata updates, important dates, supplementary material, etc.) will be announced via the "News" page of the website. All study material (i.e. study guide, online videos, supplementary workbooks, errata, etc.) can be found in the online course.

Questions

If you have a question about a particular topic in a paper or the study guide, feel free to shoot me an email at **michael@casualfellow.com**. I typically respond within 1-2 business days.

Errata

Although many hours were spent editing this study guide, errors are inevitable (especially for this initial launch). As you notice them, please email me at **michael@casualfellow.com**. An errata sheet with material errata will be posted on the online course and will be updated on an as needed basis.

Blank Pages

Since many students want a printed copy of the study guide, blank pages have been inserted throughout the guide to ensure that all outlines start on odd pages.

Bookmarks

Bookmarks have been added for each section listed in the table of contents for easier navigation in Adobe Acrobat.

BKM Ch. 6

Outline

I. Risk, Speculation and Gambling

- **Speculation** is the assumption of considerable investment risk to obtain commensurate gain
- **Gambling** is to bet or wager on an uncertain outcome

The **key difference** between speculation and gambling is that gambling lacks a "commensurate gain." For example, if a risky investment has a risk premium equal to zero (known as a **fair game**), then it is a gamble because there is no expected gain to compensate for the risk. To turn a gamble into a speculative venture, there must be an adequate risk premium present.

It's possible for a gamble to appear speculative. For **example**, suppose we have the following situation:

- Investor A will pay Investor B $100 if the value of £1 is greater than $1.40 one year from now
- Investor B will pay Investor A $100 if the value of £1 is less than $1.40 one year from now
- Thus, there are two possible outcomes – the value of £1 is greater than $1.40 one year from now OR the value of £1 is less than $1.40 one year from now
- If each investor believes that the two outcomes have equal probabilities of occurring, the expected profit to both investors is zero. This means each investor is gambling
- If each investor assigns different probabilities for each outcome, the expected profit to both investors is non-zero. This means each investor is speculating

II. Risk Aversion and Utility Values

Risk-averse investors consider only risk-free or speculative prospects with positive risk premiums. Risk-averse investors also penalize the expected rate of return of a risky portfolio. As the risk increases, the penalty increases.

How do investors choose between portfolios of varying risk? Furthermore, how do investors quantify the rate at which they are willing to trade off return against risk? To answer these questions, we will assume the following:

- ◇ Each investor assigns a **utility** score to various portfolios based on the expected return and risk of those portfolios
- ◇ Higher utility scores are given to portfolios with more attractive risk-return profiles
- ◇ The utility score is defined as follows:

$$\boxed{U = E(r) - \frac{1}{2}A\sigma^2}$$

where U is the utility score, $E(r)$ is the expected return for a portfolio, A is the investor's risk aversion and σ^2 is the variance of returns for the portfolio

Notice that utility increases as the expected return increases and decreases as the variance (i.e. risk) increases. Also, notice that A essentially controls the impact of the variance on utility. More risk-averse investors have larger values of A, which represents the higher penalty they place on risky investments. When choosing portfolios, investors attempt to maximize utility.

Example

Three investors with risk aversion indices of $A_1 = 2$, $A_2 = 3.5$ and $A_3 = 5$ are choosing between the following three portfolios:

Portfolio	Expected Return	Risk (SD)
L (low risk)	7%	5%
M (medium risk)	9	10
H (high risk)	13	20

We are also told that the risk-free rate is 5%.

The utility scores for each portfolio by investors is as follows:

Investor Risk Aversion	Utility Score of Portfolio L	Utility Score of Portfolio M	Utility Score of Portfolio H
2.0	$0.0675 = 0.07 - \frac{1}{2}(2)(0.05^2)$	0.08	**0.09**
3.5	0.0656	**0.0725**	0.06
5.0	0.0638	**0.0650**	0.03

What do we notice in the table above?

- ◇ First, let's note that a risk-free investment (i.e. $\sigma^2 = 0$) would produce a utility score of 0.05 for each investor. Thus, any investment in a risky portfolio must produce a utility score above 0.05 to be chosen
- ◇ The high-risk investor (i.e. small A) chooses Portfolio H. This is not surprising given that it doesn't penalize the large variance of Portfolio H as much as the other two investors

- ◇ The low- and medium-risk investors choose Portfolio M. Even the most risk-averse investor chooses Portfolio M over Portfolio L

The utility score is also known as the **certainty equivalent rate** of return. The certainty equivalent rate is the rate that a risk-free investment would need to offer to provide the same utility score as the risky portfolio. A portfolio is **only desirable if its certainty equivalent rate** is greater than the risk-free rate. This is the same thing we said in the first bullet above.

Risk-neutral investors with $A = 0$ ignore risk and choose portfolios based entirely on the expected return. In other words, risk-neutral investors do not penalize risk.

Risk lovers with $A < 0$ adjust the expected return upward as risk increases. In other words, they do not penalize risk. They actually consider it an enjoyable benefit.

We can depict the relationship between risk and return by plotting portfolios on a graph with axes measuring the expected return (y-axis) and the standard deviation of return (x-axis). Portfolios with equal utility live on a special curve within the graph known as an **indifference curve**. On this curve, all portfolios are equally preferred for a particular investor.

Using our example above, let's plot the indifference curves for Portfolio L for two investors: $A_1 = 2.0$ and $A_2 = 3.5$. To determine the (x, y) points for the indifference curves, we set up a table of standard deviations and solve for the expected returns that produce a utility equal to Portfolio L's utility for each investor. For the first investor ($A_1 = 2.0$), the utility (or certainty equivalent rate) was 0.0675. For the second investor ($A_2 = 3.5$), the utility was 0.0656. Then, the (x, y) points are as follows:

	Expected Return ($E[r_P]$)	
σ_P	Investor 1	Investor 2
0.00	6.75%	6.56%
0.02	6.79%	6.63%
0.04	6.91%	6.84%
0.05	7.00%	7.00%
0.06	7.11%	7.19%
0.08	7.39%	7.68%

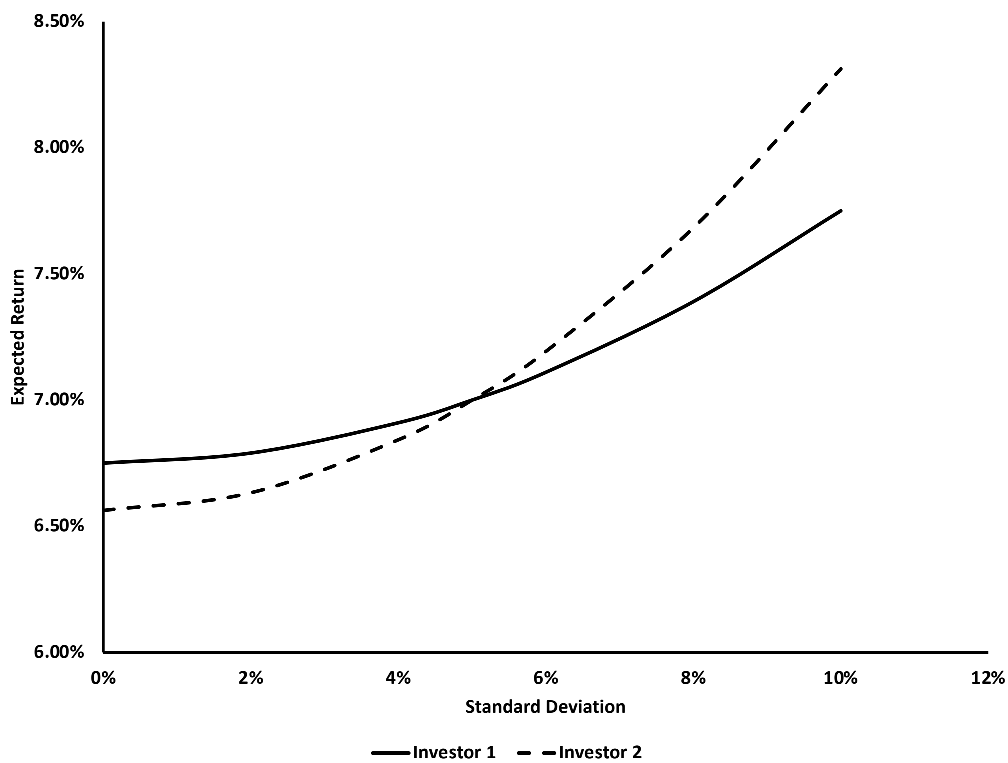
For a portfolio P, let's look at the calculation for the expected return $E[r_P]$ when the standard deviation of the returns $\sigma_P = 0.02$:

- ◇ Investor 1
  - We want Utility $= 0.0675 = E[r_P] - \frac{1}{2}(2)(0.02^2)$. Then, $E[r_P] = 0.0679$. This means that for Investor 1, a portfolio with $E[r_P] = 0.0679$ and $\sigma_P = 0.02$ is equally preferred to Portfolio L where $E[r_L] = 0.07$ and $\sigma_L = 0.05$

◇ Investor 2

- We want Utility $= 0.0656 = E[r_P] - \frac{1}{2}(3.5)(0.02^2)$. Then, $E[r_P] = 0.0663$. This means that for Investor 2, a portfolio with $E[r_P] = 0.0663$ and $\sigma_P = 0.02$ is equally preferred to Portfolio L where $E[r_L] = 0.07$ and $\sigma_L = 0.05$

The indifference curves are as follows:



What conclusions can we draw from the graph above?

◇ For the same portfolio (in this case, portfolio L), the y-intercept is lower for the investor with more risk-aversion (i.e. larger A)

◇ For the same portfolio, the y-intercepts are the utilities or certainty equivalent rates for each investor

◇ For the same portfolio, the investor with more risk-aversion will show a steeper curve. In other words, as risk increases, a more risk-averse investor requires relatively higher levels of return

◇ For the same portfolio, the indifference curves will intersect at the expected return and standard deviation of the portfolio. In this case, they intersect at (5%, 7%), which is the expected return and standard deviation for Portfolio L

In the example above, we are given A values to represent each investor's risk aversion. How might we go about estimating an investor's risk aversion?

- ◇ Design a questionnaire that allows investors to pinpoint specific levels of risk aversion coefficients
- ◇ Infer risk aversion coefficients based on empirical observations of risk and return for active investors
- ◇ Track behavior of individuals (ex. the purchase of insurance policies) to obtain average degrees of risk aversion

III. Capital Allocation Across Risky and Risk-Free Portfolios

For an investment portfolio, how much of the portfolio should be placed in risk-free securities vs. risky securities? Let P be the investor's portfolio of risky assets and F be the investor's portfolio of risk-free assets.

Let's look at an **example**. Given the following:

- ◇ The total market value of a portfolio is \$300,000
- ◇ \$90,000 of the portfolio is invested in risk-free assets
- ◇ \$210,000 of the portfolio is invested in risky assets, where \$113,400 is invested in equities (E) and \$96,600 is invested in bonds (B)

The weights (w) within the risky portfolio are as follows:

- ◇ $w_E = \frac{113{,}400}{210{,}000} = 54\%$
- ◇ $w_B = \frac{96{,}600}{210{,}000} = 46\%$

The weight of the risky portfolio, P, in total is $y = \frac{210{,}000}{300{,}000} = 70\%$. Thus, the weight of the risk-free portfolio, F, is 30%.

Suppose the owner of the portfolio would like to **decrease risk by reducing the allocation to the risky portfolio** from $y = 0.70$ to $y = 0.56$. In this case, the risky portfolio would decrease from \$210,000 to \$300,000(0.56) = \$168,000. This means that \$210,000 - \$168,000 = \$42,000 of the risky portfolio would be sold, with the proceeds used to purchase more risk-free securities. It's important to note that the weights of the risky portfolio remain unchanged after the sale. Thus, the owner would sell \$42,000(0.54) = \$22,680 of E and \$42,000(0.46) = \$19,320 of B. The risk-free portfolio would increase to \$90,000 + \$42,000 = \$132,000. We can confirm that the weights of the risk portfolio remain unchanged:

- ◇ $w_E = \frac{113{,}400-22{,}680}{210{,}000-42{,}000} = 54\%$
- ◇ $w_B = \frac{96{,}600-19{,}320}{210{,}000-42{,}000} = 46\%$

Although we have not changed the composition of the risky portfolio, we have reduced the risk of the complete portfolio by shifting to more risk-free securities.

We mentioned the composition of the risky portfolio. What about the risk-free portfolio? When we think of risk-free assets, the most common things to come to mind are **Treasury Bills**. Due to their short-term nature, they are not sensitive to interest rate fluctuations or inflation uncertainty. In addition, they are issued by the government which means they should be default-free. In practice, other Treasury and U.S. agency securities, as well as repurchase agreements, are considered risk-free. Again, this is driven by their short-term nature as well as lack of default and credit risk.

IV. Portfolios of One Risky Asset and a Risk-Free Asset

In the prior section, we discussed an example where the owner of a portfolio decreased the allocation to the risky portfolio from $y = 0.70$ to $y = 0.56$. In the next two sections, we will examine how we determine the optimal allocation to the risky portfolio.

The **expected rate of return on the complete portfolio** is as follows:

$$\boxed{\begin{aligned} E(r_C) &= yE(r_P) + (1-y)r_f \\ &= r_f + y[E(r_P) - r_f] \\ &= r_f + y \times \text{Risk Premium} \end{aligned}}$$

where $E(r_C)$ is the expected return for the complete portfolio C, $E(r_P)$ is the expected return for the risky portfolio P, y is the allocation of the complete portfolio to the risky portfolio and r_f is the risk-free asset. In the equation above, the base rate of return for any portfolio is the risk-free rate. Beyond the risk-free rate, the portfolio is expected to earn a proportion y of the risky portfolio's return over the risk-free rate.

The **standard deviation on the complete portfolio** is as follows:

$$\boxed{\sigma_C = y\sigma_P}$$

where σ_C is the standard deviation of the complete portfolio and σ_P is the standard deviation of the risky portfolio.

We can rewrite the expected return of the complete portfolio as a function of the standard deviation of the complete portfolio using the fact that $y = \frac{\sigma_C}{\sigma_P}$:

$$\boxed{\begin{aligned} E(r_C) &= r_f + y[E(r_P) - r_f] \\ &= r_f + \frac{\sigma_C}{\sigma_P}[E(r_P) - r_f] \\ &= r_f + \frac{[E(r_P) - r_f]}{\sigma_P}\sigma_C \\ &= r_f + S\sigma_C \end{aligned}}$$

where $S = \frac{[E(r_P)-r_f]}{\sigma_P}$ is the **Sharpe ratio** (also known as the reward-to-volatility ratio). In the equation above, we have shown that the expected return of the complete portfolio as a function of its standard deviation is a straight line with a y-intercept equal to the risk-free rate and a slope equal to the Sharpe Ratio. This straight line is known as the **capital allocation line (CAL)** and represents **the investment opportunity set**, which is the set of all risk-return combinations available to investors.

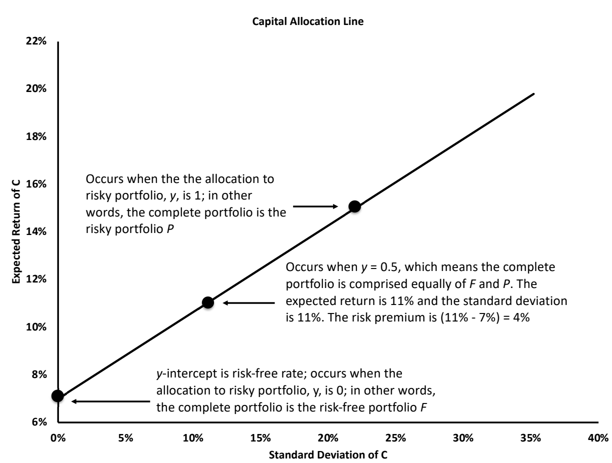
The Sharpe Ratio can be interpreted as the average percentage excess return over the risk-free rate for every 1% of additional standard deviation. For example, if the Shape Ratio is 0.40, then stock investors obtain a 0.40% average excess return over the risk-free rate for every 1% of additional standard deviation.

Example

Given the following:

- ◇ $E(r_P) = 15\%$
- ◇ $\sigma_P = 22\%$
- ◇ $r_f = 7\%$

Then, the Sharpe Ratio $S = \frac{[E(r_P)-r_f]}{\sigma_P} = \frac{0.15-0.07}{0.22} = \frac{8}{22}$. Thus, the CAL is $E(r_C) = r_f + S\sigma_C = 0.07 + \frac{8}{22}\sigma_C$. Let's go ahead and graph this line:



We notice the following in the graph above:

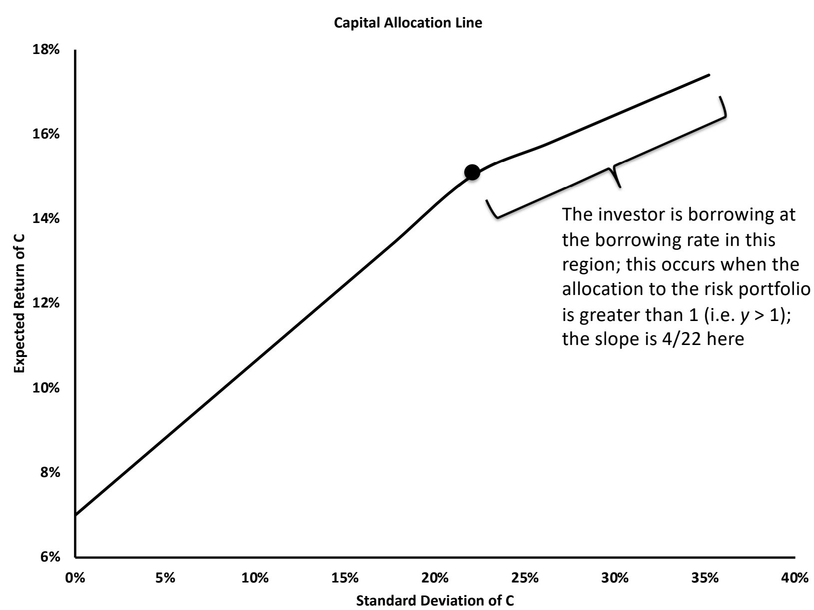
- ◇ Recall that $E(r_C)$ can be written as $E(r_C) = yE(r_P) + (1-y)r_f$. When $y = 0$, the complete portfolio consists solely of the risk-free portfolio F. This occurs at the y-intercept
- ◇ When $y = 1$, the complete portfolio consists solely of the risky portfolio P. At this point on the graph, $\sigma_C = y\sigma_P = (1)(0.22) = 22\%$ and $E(r_C) = (1)(0.15) + (1-1)(0.07) = 15\%$. Alternatively, we could have calculated $E(r_C) = r_f + S\sigma_C = 0.07 + \frac{8}{22}(0.22) = 15\%$
- ◇ When $y = 0.5$, the complete portfolio consists equally of F and P. At this point on the graph, $\sigma_C = y\sigma_P = 0.5(0.22) = 11\%$ and $E(r_C) = r_f + S\sigma_C = 0.07 + \frac{8}{22}(0.11) = 11\%$

◇ The line extends past the point where the complete portfolio consists solely of P. How is that possible? Can $y > 1$? Yes! If investors can borrow at the risk-free rate of $r_f = 7\%$, then they can construct portfolios to the right of P on the CAL. As an example:

- The market value of a risky portfolio is \$300,000
- An investor borrows an additional \$120,000
- This is a **levered position** in the risky asset since some of the financing is coming from borrowed money. In this case, $y = \frac{420{,}000}{300{,}000} = 1.4$
- Thus, the amount allocated to the risk-free asset is $1 - y = 1 - 1.4 = -.4$. This reflects the fact that investor has a borrowing position in the risk-free asset
- Since levered positions are further to the right on the CAL, they have higher standard deviations than un-levered positions. This is not surprising given the additional uncertainty

Regarding the borrowing position above, points to the right of P on the CAL assume that the investor can borrow at the risk-free rate. In reality, non-government investors cannot borrow at the risk-free rate since they have some level of default risk. Thus, a non-government borrower's rate will always be higher than the risk-free rate. This creates a "kink" on the CAL at the point where $y = 1$. Beyond this point, the slope of the CAL changes to reflect the borrower's rate.

Suppose the borrower's rate is $r_f^B = 11\%$. Then, beyond the point where $y = 1$, the Sharpe Ratio $S_{y>1} = \frac{[E(r_P) - r_f]}{\sigma_P} = \frac{0.15 - 0.11}{0.22} = \frac{4}{22}$. This creates the following graph:



To calculate the "kinked" portion of the CAL, we use $\sigma_C = y\sigma_P$ and $E(r_C) = r_f^B + S_{y>1}\sigma_C$. Notice that these are the same formulas as above. However, we have replaced r_f with r_f^B and

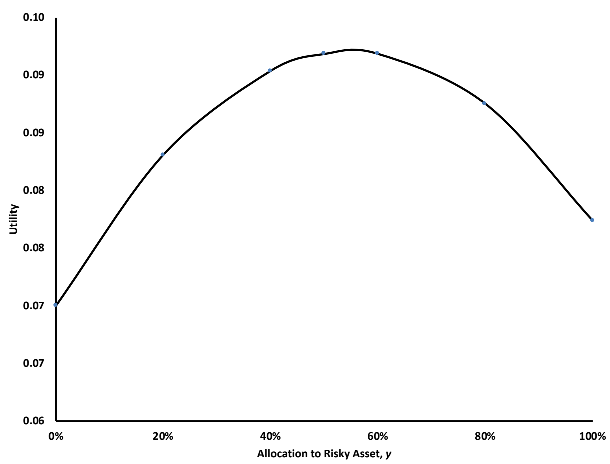
S with $S_{y>1}$. For **example**, suppose $y = 1.2$. Then, $\sigma_C = y\sigma_P = 1.2(0.22) = 26.4\%$ and $E(r_C) = r_f^B + S_{y>1}\sigma_C = 0.11 + \frac{4}{22}(0.264) = 15.8\%$.

V. Risk Tolerance and Asset Allocation

This section will discuss how an investor determines where on the CAL to reside. For any investor, the goal is to choose the portfolio that maximizes utility. Recall that utility $U = E(r) - \frac{1}{2}A\sigma^2$. Let's consider an example:

- $E(r_P) = 15\%$
- $\sigma_P = 22\%$
- $r_f = 7\%$
- $A = 3$

Notice that this is the same setup as our example in the prior section. Let's plot the utility U as a function of the allocation y to the risky portfolio P:



The graph above shows that utility increases with y up until a certain point. Then, the volatility outweighs the expected return causing utility to decrease.

We want to choose the y that corresponds to the maximum point on the utility curve. As you might have imagined, this is a calculus problem. We need to take the derivative of $U = E(r) - \frac{1}{2}A\sigma^2$ with respect to y and set it equal to 0. Using our formulas from the prior section, we can rewrite U as $U = r_f + y[E(r_P) - r_f] - \frac{1}{2}Ay^2\sigma_P^2$. The derivative of U with respect to y is as follows:

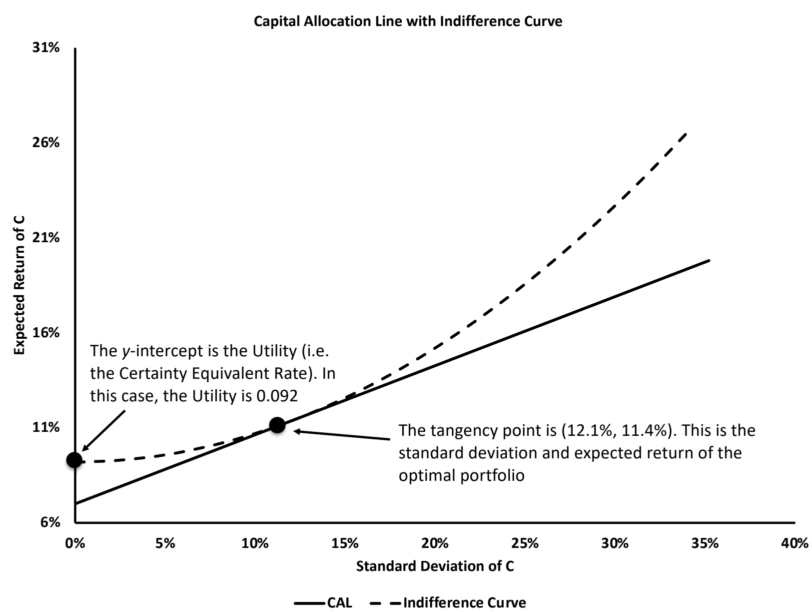
$$\frac{dU}{dy} = E(r_P) - r_f - Ay\sigma_P^2$$

If we set $E(r_P) - r_f - Ay\sigma_P^2 = 0$, we find that the optimal position for risk-averse investors in the risky asset is as follows:

$$\boxed{y^* = \frac{E(r_P) - r_f}{A\sigma_P^2}}$$

Using our **example**, we find that $y^* = \frac{0.15-0.07}{3(0.22^2)} = 0.55$. Thus, to maximize utility, an investor with $A = 3$ should allocate 55% to the risky portfolio P. The utility at this allocation level is $U = r_f + y[E(r_P) - r_f] - \frac{1}{2}Ay^2\sigma_P^2 = 0.07 + 0.55(0.15 - 0.07) - \frac{1}{2}(3)(0.55^2)(0.22^2) = 0.092$. The expected return is $E(r_C) = yE(r_P) + (1-y)r_f = 0.55(0.15) + (1-0.55)(0.07) = 11.4\%$, and the standard deviation is $\sigma_C = y\sigma_P = 0.55(0.22) = 12.1\%$.

Another way to find y^* is using indifference curves. Recall from earlier that indifference curves plot all risk-return combinations that produce the same utility for a particular investor. For the same investor, higher indifference curves (i.e. higher y-intercepts) are better because they represent higher utilities. The **key result** is that the **highest indifference curve that still touches the CAL** (i.e. tangent to the CAL) **identifies the optimal complete portfolio**. In particular, the tangency point **corresponds to the standard deviation and expected return of the optimal complete portfolio**. Here is the capital allocation line from our example, along with the highest indifference curve that touches the CAL:



In the graph above, the y-intercept of the indifference curve is 0.092. The tangency point is $(12.1\%, 11.4\%)$. These are the same figures we arrived at above for U, σ_C and $E(r_C)$.

The last thing to note from the graph above is that the choice for y^* is determined by **two things**:

(1) Risk Aversion – This is captured by the slope of the indifference curve. Higher risk aversion (i.e. larger A) results in a steeper indifference curve

(2) Sharpe Ratio – This is captured by the slope of the opportunity set or CAL. A different Sharpe Ratio impacts the tangency point of the highest indifference curve

There is one more thing to note from this section. In all of the preceding discussion, we have treated standard deviation as the appropriate measure of risk. This **implicitly assumes that returns are normally distributed**. However, this is not necessarily the case. In cases of non-normality with fat tails, investors should consider measuring the Value at Risk (VaR) and reducing their allocation to the risky portfolio.

VI. Passive Strategies: The Capital Market Line

A **passive strategy** avoids any direct or indirect security analysis. It involves investment in two portfolios:

(1) Risk-free short-term T-bills (or alternatively, a money market fund)

(2) A well-diversified portfolio of common stocks that acts similarly to a broad market index. This is known as a "neutral" diversification strategy. Under this approach, if a stock comprises 3% of the market value of all listed stocks, then it should comprise 3% of the investor's risky portfolio

The capital allocation line that results from holding a mix of the risk-free asset and a broad index of common stocks is known as the **capital market line**. This is the line produced by a passive strategy.

Why might one pursue a passive strategy over an active strategy?

- ◇ Passive strategies are less expensive than active strategies
  - If an investor pursues an active strategy on his or her own, there is time and cost to acquire the information needed to generate an optimal active portfolio of risky assets. If an investor hires a professional to pursue an active strategy, there is the fee owed to the professional
  - Passive strategies only require the small costs to purchase T-bills and modest fees to the management company
- ◇ Passive strategies enjoy the free-rider benefit
  - When there are a number of active investors, they quickly bid up prices of undervalued assets (through buying) and force down prices of overvalued assets (through selling). This leads to most assets being fairly priced at any given time
  - This means that a well-diversified portfolio of common stock should be a fair buy, resulting in a passive strategy that may not be any worse than an active strategy

BKM Ch. 7

Outline

I. Diversification and Portfolio Risk

Individual stocks are subject to two sources of risk:

(1) Nonsystematic risk

- ◇ Refers to risk that is specific to a single firm
- ◇ This risk can be effectively reduced to zero by diversifying a portfolio
- ◇ Diversification occurs by increasing the number of securities in a portfolio and spreading risk exposure over independent, firm-specific risk sources. This same idea underlies the **insurance principle**. When an insurer writes a large number of independent policies (i.e. independent sources of risk), diversification occurs
- ◇ Portfolio volatility should decrease as the number of securities in the portfolio increases
- ◇ This risk is also known as unique risk, firm-specific risk or diversifiable risk

(2) Systematic risk

- ◇ Refers to risk that is common to all stocks. This includes inflation, interest rate risk and exchange rate risk
- ◇ This risk cannot be diversified away
- ◇ This risk is also known as market risk or nondiversifiable risk

II. Portfolios of Two Risky Assets

In this section, we will build a risky portfolio that minimizes risk for an expected rate of return. To demonstrate this concept, we will look at a portfolio P of two risky assets. The first risky asset represents a bond portfolio D and the second risky asset represents an equity asset E. The expected rate of return for the portfolio is defined as follows:

$$\boxed{E(r_P) = w_D E(r_D) + w_E E(r_E)}$$

where r_P is the total portfolio return, r_D is the bond asset return, r_E is the equity asset return, w_D is the proportion of the total portfolio invested in the bond asset and w_E is the proportion of the total portfolio invested in the equity asset.

The variance of the portfolio is defined as follows:

$$\boxed{\begin{aligned}\sigma_P^2 &= w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_Dw_ECov(r_D, r_E)\\ &= w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_Dw_E\sigma_D\sigma_E\rho_{DE}\end{aligned}}$$

where σ_P^2 is the variance of the total portfolio, σ_D^2 is the variance of the bond asset, σ_E^2 is the variance of the equity asset and ρ_{DE} is the correlation between the bond and equity assets.

As a **quick aside**, we can **extend the two asset portfolio to three assets**. Suppose the risky portfolio consists of assets X, Y and Z. Then, the expected rate of return and variance are as follows::

- ⋄ $E(r_P) = w_XE(r_X) + w_YE(r_Y) + w_ZE(r_Z)$
- ⋄ $\sigma_P^2 = w_X^2\sigma_X^2+w_Y^2\sigma_Y^2+w_Z^2\sigma_Z^2+2w_Xw_YCov(r_X, r_Y)+2w_Xw_ZCov(r_X, r_Z)+2w_Yw_ZCov(r_Y, r_Z)$
- ⋄ Alternatively, we can write the variance as $\sigma_P^2 = w_X^2\sigma_X^2+w_Y^2\sigma_Y^2+w_Z^2\sigma_Z^2+2w_Xw_Y\sigma_X\sigma_Y\rho_{XY}+2w_Xw_Z\sigma_X\sigma_Z\rho_{XZ} + 2w_Yw_Z\sigma_Y\sigma_Z\rho_{YZ}$

Now, back to our two-asset risky portfolio. There are some **key takeaways** from the formulas above:

- ⋄ The expected rate of return of the total portfolio is a weighted average of the expected rates of return for the individual assets
- ⋄ When $\rho_{DE} = 1$, the standard deviation of the total portfolio is a weighted average of the standard deviations for the individual assets
- ⋄ When $\rho_{DE} \neq 1$, the standard deviation of the total portfolio is LESS THAN the weighted average of the standard deviations for the individual assets. This is the diversification benefit
- ⋄ Assets that have negative correlations with other assets are known as **hedge assets**

Given that the variance of the portfolio represents risk, it makes sense to select weights that minimize the variance of the portfolio. This is known as the **minimum-variance portfolio**. The weights are as follows:

$$\boxed{\begin{aligned}&w_{\text{Min}}(D) = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)}\\ &w_{\text{Min}}(E) = 1 - w_{\text{Min}}(D)\end{aligned}}$$

When $\rho_{DE} = 1$, there is no diversification benefit. Thus, the minimum-variance portfolio consists only of the asset with the lowest standard deviation.

When $\rho_{DE} = -1$, a perfect hedge exists. In this case, we can produce a portfolio with a standard deviation equal to zero. The weights are as follows:

$$w_{\text{Min}}(D) = \frac{\sigma_E}{\sigma_D + \sigma_E}$$
$$w_{\text{Min}}(E) = \frac{\sigma_D}{\sigma_D + \sigma_E} = 1 - w_{\text{Min}}(D)$$

These are the same weights that one would obtain by plugging $\rho DE = -1$ into the covariance terms in the minimum-variance portfolio formulas above.

As one might expect, the variance of bond assets is lower than the variance of equity assets. How does the portfolio standard deviation change as we increase the proportion of our investments in equities?

- ◇ For low correlations between the bond and equity assets, the portfolio standard deviation initially decreases as we move our investments out of bonds and into equities. This is due to diversification. As the portfolio becomes weighted more heavily towards equities, the increased volatility from the equity asset causes the portfolio standard deviation to increase. This produces a U-shape
- ◇ For high correlations between the bond and equity assets, the portfolio standard deviation increases monotonically as we move our investments out of bonds and into equities

III. Asset Allocation with Stocks, Bonds and Bills

Recall from Ch. 6 that the Sharpe Ratio is defined as a portfolio's risk premium in excess of the risk-free rate divided by the portfolio's standard deviation. Thus, higher Sharpe Ratios are preferable because they offer larger rewards relative to volatility. Maximizing the Sharpe Ratio for the risky portfolio is equivalent to choosing the risky portfolio that produces the steepest capital allocation line (CAL) (recall that the Sharpe Ratio is the slope of the CAL). Please note that the **Sharpe Ratio of the complete portfolio is equal to the Sharpe Ratio of the risky portfolio**. Both portfolios live on the CAL. Hence, both portfolios have the same Sharpe Ratios since the slope of the CAL is the same regardless of where you live on the line (except for the possibility of a "kinked" section due to borrowing money).

For a risky portfolio of two assets, the **portfolio opportunity set** represents all combinations of portfolio expected returns and standard deviations that can be constructed between the assets. The goal is to find the CAL that is tangent to the portfolio opportunity set. This is the maximum Sharpe Ratio (i.e. steepest CAL) possible for those two assets. Using calculus, we find that the **optimal risky portfolio** is given by the following weights:

$$\boxed{\begin{array}{l} w_D = \dfrac{E(R_D)\sigma_E^2 - E(R_E)Cov(R_D, R_E)}{E(R_D)\sigma_E^2 + E(R_E)\sigma_D^2 - [E(R_D) + E(R_E)]Cov(R_D, R_E)} \\ w_E = 1 - w_D \end{array}}$$

where R represents the **excess returns over the risk-free rate** rather than the total returns denoted by r. Note that the optimal risky portfolio is not necessarily equal to the minimum-variance portfolio. The idea behind the optimal risky portfolio is to maximize the Sharpe Ratio. The idea behind the minimum-variance portfolio is to minimize the variance. These are two different goals which often lead to two different portfolios.

Once we have found the optimal risky portfolio, we can use our Ch. 6 methodology to find the **optimal complete portfolio**.

Example

Suppose an investor is building a portfolio of stock, bond and risk-free assets. Given the following:

Asset	Expected Return	Standard Deviation
Stocks	9%	21%
Bonds	3%	12%
T-Bills	1%	0%

- $\rho_{DE} = 0.2$
- $A = 3$

First, let's determine and characterize the optimal risky portfolio:

- $w_D = \frac{E(R_D)\sigma_E^2 - E(R_E)Cov(R_D,R_E)}{E(R_D)\sigma_E^2 + E(R_E)\sigma_D^2 - [E(R_D)+E(R_E)]Cov(R_D,R_E)}$
- Recall that R represents excess returns over the risk-free rate. Thus, $E(R_D) = 0.03 - 0.01 = 0.02$ and $E(R_E) = 0.09 - 0.01 = 0.08$
- $Cov(R_D, R_E) = \rho_{DE}\sigma_D\sigma_E = 0.2(0.12)(0.21) = 0.00504$
- $w_D = \frac{0.02(0.21^2) - 0.08(0.00504)}{0.02(0.21^2) + 0.08(0.12^2) - (0.02+0.08)(0.00504)} = 0.313$
- $w_E = 1 - 0.313 = 0.687$
- The expected return of the optimal risky portfolio is $E(r_P) = w_D E(r_D) + w_E E(r_E) = 0.313(0.03) + (0.687)(0.09) = 7.12\%$
- The variance of the optimal risky portfolio is $\sigma_P^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E Cov(r_D, r_E) = 0.313^2(0.12^2) + 0.687^2(0.21^2) + 2(0.313)(0.687)(0.00504) = 2.44\%$
- The standard deviation of the optimal risky portfolio is $\sqrt{\sigma_P^2} = \sqrt{0.0244} = 15.62\%$
- The Sharpe Ratio of the optimal risky portfolio is $S = \frac{E(r_P) - r_f}{\sigma_P} = \frac{0.0712 - 0.01}{0.1562} = 0.392$. This is the maximum possible Sharpe Ratio and the slope of the steepest CAL between the two risky assets

Second, let's determine and characterize the optimal complete portfolio:

- The optimal amount to allocate to the risky portfolio is $y^* = \frac{E(r_P)-r_f}{A\sigma_P^2} = \frac{0.0712-0.01}{3(0.0244)} = 83.6\%$
- The expected return of the optimal complete portfolio is $E(r_C) = (1-y^*)(r_f) + y^*E(r_P) = (1-0.836)(0.01) + (0.836)(0.0712) = 6.12\%$
- The standard deviation of the optimal complete portfolio is $\sigma_C = y^*\sigma_P = (0.836)(0.1562) = 13.06\%$
- The utility of the optimal complete portfolio is $U = E(r_C) - \frac{1}{2}A\sigma_C^2 = 0.0612 - \frac{1}{2}(3)(0.1306^2) = 0.0356$. This is the maximum possible utility between the risk-free asset and the optimal risky portfolio. This portfolio is also the point on the graph (as defined by the standard deviation and expected return) where the highest indifference curve for this investor occurs that is still tangent to the CAL (remember that higher difference curves represent higher utilities). See Chapter 6 for a reminder

Third, let's determine the final allocation of each asset within the optimal complete portfolio:

- Stocks = $y^*(w_E) = 0.836(0.687) = 57.43\%$
- Bonds = $y^*(w_D) = 0.836(0.313) = 26.17\%$
- T-Bills = 1 - 0.5744 - 0.2617 = 16.40%

Lastly, let's determine the minimum-variance risky portfolio weights:

- $w_{\text{Min}}(D) = \frac{\sigma_E^2 - Cov(\sigma_D,\sigma_E)}{\sigma_D^2+\sigma_E^2-2Cov(r_D,r_E)} = \frac{0.21^2-0.00504}{0.12^2+0.21^2-2(0.00504)} = 80.7\%$
- $w_{\text{Min}}(E) = 1 - w_{\text{Min}}(D) = 1 - 0.807 = 19.3\%$
- Notice that these weights differ from the optimal risky portfolio quite a bit. The minimum-variance risky portfolio allocates far more of the investment to the lower-risk bond assets. Although the minimum-variance risky portfolio produces a lower variance than the optimal risky portfolio, it produces a smaller Sharpe Ratio. This makes it less attractive in terms of reward-to-risk

Before moving on to the next section, I want to highlight a couple of oddities that might arise when building the optimal risky portfolio. We covered this in Ch. 6, but I think it's helpful to look at it again:

- Suppose the optimal risky portfolio produces $w_D > 1$. This means that the investor should sell the equity asset short and invest the proceeds of the short sale into the bond asset
- Suppose the optimal risky portfolio produces $w_E > 1$. This means that the investor should sell the bond asset short and invest the proceeds of the short sale into the equity asset

◇ *I can imagine an exam problem where the final weight for one of the assets is greater than one and you are asked to interpret. Make sure you know what it means when weights are greater than one*

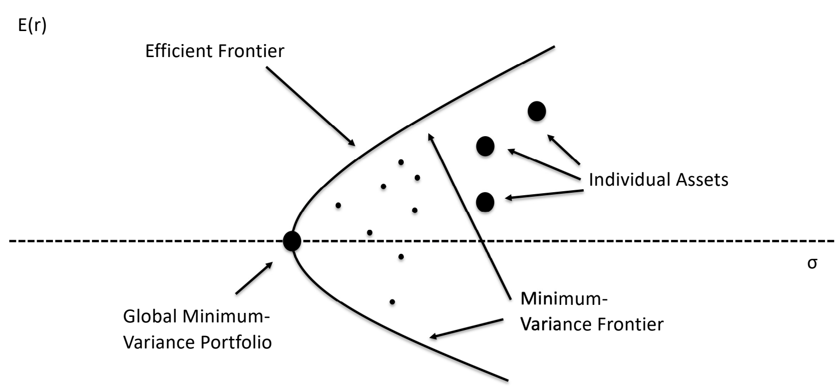
IV. The Markowitz Portfolio Optimization Model

In this section, we will generalize the two asset scenario to many assets. Like the two asset scenario, the textbook breaks this process into **three steps**:

(1) Determine the available risk-return opportunities to the investor

◇ Unlike the two asset example, this cannot be represented by a single portfolio opportunity set as there are an infinite number of ways to combine more than two stocks

◇ However, we can still build an outer frontier, which is similar to the portfolio opportunity set. The outer frontier is known as the **minimum-variance frontier**, and it graphs the minimum variance portfolio for each given expected return:



The global minimum-variance portfolio represents the portfolio with the smallest variance among all possible portfolios that can be constructed from risky assets. All portfolios ON the minimum-variance frontier AND BELOW the global minimum-variance portfolio are NOT efficient since there is a portfolio with the same standard deviation and higher expected return directly above it. All portfolios ON the minimum-variance frontier AND ABOVE the global minimum-variance portfolio are efficient because they represent the portfolios with the highest expected return for each standard deviation (i.e. each level of risk). Notice that individual assets lie to the right of the minimum-variance frontier. This means that portfolios containing single risky assets are inefficient. Diversification allows us to construct portfolios with higher expected returns and lower standard deviations

(2) Find the CAL with the highest Sharpe Ratio

  - ◇ Like the two asset scenario, the CAL with the highest Sharpe Ratio is tangent to the efficient frontier
  - ◇ The tangency point identifies the optimal risky portfolio

(3) Find the optimal mix between the optimal risky portfolio and the risk-free asset

Let's look at each step in more detail.

Step 1: Determine the available risk-return opportunities to the investor

Step 1 involves the following sub-steps:

- ◇ Produce a set of expected returns, variances and covariance estimates corresponding to the universe of n risky securities. This is known as the input list
- ◇ Feed the input list into an optimization routine to find the efficient frontier. This is the top half of the minimum-variance frontier (i.e. the part above the global minimum variance portfolio). The optimization routine can look for the lowest standard deviation for each expected rate of return or the highest expected rate of return for each standard deviation. Either way, the optimization routine will produce a number of efficient portfolios which ultimately produce the efficient frontier

Some other things to consider during the optimization routine are as follows:

- ◇ If shorting is prohibited, we may want to constrain the optimization routine to exclude negative positions in securities. In this case, it's possible for single assets to represent efficient portfolios
- ◇ We may want to consider only portfolios that exceed a minimum dividend yield
- ◇ We may want to exclude any investments in ethically or politically controversial industries

Steps 2 & 3: Find the CAL with the highest Sharpe Ratio AND identify the optimal mix between the optimal risky portfolio and the risk-free asset

We can find the CAL with the highest Sharpe Ratio in two different ways:

(1) Plot the CAL for various efficient, risky portfolios until we find the one that is tangent to the efficient frontier

(2) Maximize the Sharpe Ratio directly using an optimization routine within a spreadsheet

Interestingly, the optimal risky portfolio is the same for ALL INVESTORS regardless of risk aversion. Notice that we have not considered "A" yet. This only comes into play when we are determining the amounts to allocate to the risk-free asset and the optimal risky portfolio. An investor with more risk-aversion (i.e. higher "A") will allocate more to the risk-free asset and shift to the left on the CAL. An investor with less risk-aversion (i.e. lower "A") will allocate more to the optimal risky portfolio and shift to the right on the CAL. This is known as the **separation property**. The separation property says that portfolio choice comes down to two independent tasks:

(1) Determine the optimal risky portfolio. This is an objective exercise and should result in the same portfolio for every investor

(2) Determine the proper capital allocation (i.e. the amount to allocate to the risk-free asset vs. the optimal risky portfolio) based on the investor's risk aversion. This is a subjective exercise and often results in a different portfolio for each investor

Although the optimal risky portfolio *should* be the same for all investors, that is often not the case in practice. Different portfolio managers typically estimate different input lists. As the input lists change, the efficient frontiers of risky portfolios change. Some input lists are better than others. If an input list consists of mis-priced securities, then the optimal risky portfolio produced by that list will underperform.

V. The Power of Diversification

In a prior section, we showed the variances of two-asset and three-asset portfolios. We can extend this to n-asset portfolios as follows:

$$\sigma_P^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j Cov(r_i, r_j)$$

If we hold an equally-weighted portfolio where $w_i = \frac{1}{n}$ for each security, then σ_P^2 can be rewritten as follows:

$$\sigma_P^2 = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{n}\sigma_i^2 + \sum_{\substack{j=1 \\ j\neq i}}^{n}\sum_{i=1}^{n}\frac{1}{n^2}Cov(r_i, r_j)$$

This can be further simplified to the following:

$$\boxed{\sigma_P^2 = \frac{1}{n}\overline{\sigma}^2 + \frac{n-1}{n}\overline{Cov}}$$

where $\overline{\sigma}^2 = \frac{1}{n}\sum\limits_{i=1}^{n}\sigma_i^2$ and $\overline{Cov} = \frac{1}{n(n-1)}\sum_{\substack{j=1 \\ j\neq i}}^{n}\sum\limits_{i=1}^{n} Cov(r_i, r_j)$. These two quantities represent the average variance and average covariance of the n securities. If we **assume that all securities share the same standard deviation $\boldsymbol{\sigma}$ and correlation coefficient $\boldsymbol{\rho}$**, then the above formula simplifies to the following:

$$\boxed{\sigma_P^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2}$$

The **first term** above represents **firm-specific risk**. The **second term** above represents **systematic risk**. We can see the effects of diversification for different levels of correlation:

- ◇ When $\rho = 0$, systematic risk goes to zero. As diversification increases (i.e n gets larger), then the first term approaches zero. Thus, when $\rho = 0$, we can use diversification to effectively reduce all volatility

- ◇ When $\rho = 1$, $\sigma_P^2 = \sigma^2$. Thus, under perfect correlation, diversification has no benefit and all risk is systematic

- ◇ In general terms, as n increases, σ_P^2 approaches $\rho\sigma^2$

- ◇ Note that the textbook specifically defines systematic risk as $\frac{n-1}{n}\overline{Cov}$ as n approaches infinity (see the solution to Concept Check 7.5 for more details). As n approaches infinity, this simplifies to the average covariance $\overline{Cov}$. Any residual variance comes from the firm-specific risk

The **main takeaway** regarding **diversified portfolios** is that an individual security's contribution to the portfolio risk depends on its covariance with the other securities in the portfolio. It does not depend on the individual security's variance since the firm-specific risk tends to zero for well-diversified portfolios.

Example

Suppose we have a universe of 35 risky securities. Further assume that returns are identically distributed with an expected return of 8% and a standard deviation of 25%. The securities share a common correlation of 0.10.

First, let's calculate the total variance of the portfolio:

- ◇ $\sigma_P^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2 = \frac{1}{35}(0.25^2) + \frac{34}{35}(0.10)(0.25^2) = 0.79\%$

Second, let's calculate the systematic risk and firm-specific risk:

- ◇ Systematic risk is calculated as the average covariance $\overline{Cov}$. Since the securities share the same standard deviation and correlation coefficient, the average covariance is simply $\rho\sigma^2 = 0.10(0.25^2) = 0.63\%$

- ◇ Firm-specific risk is Total Portfolio Variance – Systematic Risk = 0.79% – 0.63% = 0.16%

Example

Suppose we wanted to produce an efficient portfolio with a standard deviation less than 8%. We can solve for the smallest number of stocks necessary to produce such a portfolio. We want $\sigma_P^2 =$

$\frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2 < (0.08^2)$. Thus:

$$
\begin{aligned}
(0.08^2) &> \frac{1}{n}(0.25^2) + \frac{n-1}{n}(0.10)(0.25^2) \\
(0.08^2) &> (0.25^2)\left[\frac{1}{n} + \frac{0.10n - 0.10}{n}\right] \\
0.1024 &> \frac{0.10n + 0.90}{n} \\
0.1024n - 0.10n &> 0.90 \\
0.0024n &> 0.90 \\
n &> 375
\end{aligned}
$$

We need at least 375 stocks in our portfolio to produce a standard deviation less than 8%.

At the end of this section, the authors provide **three reasons** why investors distinguish between asset allocation and security selection (even though the two concepts essentially do the same thing):

(1) The demand for investment management has increased over time

(2) Financial markets have become too sophisticated for amateur investors

(3) There are economies of scale in investment analysis

As investments have become broader and more global, more specialized expertise is required. Thus, each separate asset class must be managed independently. This makes it impossible to optimize an organization's risky portfolio in one stage. Instead, the security selection of each asset class must be optimized independently first. Then, the allocation to each asset class is updated constantly based on the organization's investment budget.

VI. Risk Pooling, Risk Sharing and the Risk of Long-Term Investments

As mentioned earlier, investors use diversification to reduce risk. This risk reduction relies on both risk pooling and risk sharing.

Risk pooling is defined as adding uncorrelated, risky projects to an investor's portfolio. We referred to this earlier as the insurance principle. It turns out that risk pooling by itself does not reduce risk. To help drive this home, let's consider an **example**:

- Suppose an investor invests \$1 in Asset A and \$1 in Asset B
- The rate of return for Asset A is r_A and the rate of return for Asset B is r_B. The expected rate of return for each Asset is $E(r)$
- The variance of the returns for each asset is σ^2
- The assets are independent

Using our formulas for two-asset portfolios (with a covariance equal to zero since they are independent), we find that $E(r_P) = \frac{1}{2}E(r) + \frac{1}{2}E(r) = E(r)$ and $Var(r_P) = (\frac{1}{2})^2\sigma^2 + (\frac{1}{2})^2\sigma^2 = (\frac{1}{2})\sigma^2$. This suggests that the two-asset portfolio is safer because its variance is half of the variance of a one-asset portfolio. However, this only applies to the RATE of return. For the actual dollar return, the variance doubled:

- ⋄ The dollar profits of each asset are r_A and r_B, respectively
- ⋄ Thus, the variance of the total dollar profit is $Var(r_A + r_B) = 2\sigma^2$

The reason why the results above conflict is because we are comparing two portfolios of different sizes. One is a two-asset portfolio equal to $2. The other is a one-asset portfolio equal to $1. The two-asset portfolio may have more predictable rates of return. However, its dollar return is less predictable because the portfolio is larger. This is where risk sharing comes in.

Risk sharing is defined as allowing other investors to share in the risk of the asset portfolio. Back to our **example**:

- ⋄ Suppose an investor invests $1 in Asset A and $1 in Asset B
- ⋄ Suppose the same investor sells half of his total investment to other investors

In this case, his portfolio still consists of two assets. However, the amount of the portfolio is only $1. Now we can directly compare a one-asset portfolio to the two-asset portfolio because they are equal in size. We no longer have to consider the dollar returns because the sizes are no longer different. Instead, we can simply compare the rates of return:

- ⋄ Just as we showed above, $Var(r_P) = (\frac{1}{2})\sigma^2$
- ⋄ This now represents a true reduction in risk because the investment was fixed at $1

We can relate this to capital markets as follows. Investors reduce risk by adding more stocks to a risky portfolio **while leaving the total size of the portfolio unchanged**. This captures both risk pooling (adding uncorrelated assets) and risk sharing (leaving the total size unchanged). By risk sharing, the investor reduces his/her exposure to an individual stock and its firm-specific risk.

Next, the authors go into a quick aside on the relationship between diversification and the Sharpe Ratio. Here are the highlights based on our prior example:

- ⋄ Asset A has a Sharpe Ratio of $S_A = \frac{E(r)-r_f}{\sigma}$
- ⋄ Using risk pooling only, the two-asset portfolio with a size of $2 has a Sharpe Ratio of $S_P = \frac{\text{Expected dollar risk premium}}{SD(\text{Dollar Profit})} = \frac{2[E(r)-r_f]}{\sqrt{2\sigma^2}} = \sqrt{2}S_A$

- ◇ Combining risk pooling with risk sharing, the two-asset portfolio with a size of \$1 has a Sharpe Ratio of $S_P = \frac{\text{Expected dollar risk premium}}{SD(\text{Dollar Profit})} = \frac{[E(r)-r_f]}{\sqrt{\frac{1}{2}\sigma^2}} = \sqrt{2}S_A$

Although it does not reduce risk, risk pooling does improve the Sharpe Ratio. However, when combined with risk sharing, we see the same improvement in the Sharpe Ratio along with risk reduction.

The last thing covered in this section is time diversification. Some investors mistakenly believe that risk is reduced by spreading investments across time. This is not the case. Choosing to invest in an additional time period is similar to choosing to invest in an additional risky asset without fixing the initial investment. It doubles the variance.

As an **example**, suppose we have an investment budget that must be applied over a two-year period. Here are three ways to implement that investment budget:

(1) Invest the entire budget in a risky asset for the first year. Then, move the entire investment budget into a risk-free asset for the second year. Since the investment budget was only subject to the risky asset for a year, the characteristics of the investment over the two-year horizon are as follows:

- ◇ Expected risk premium $= E(r) - r_f$
- ◇ SD $= \sigma$
- ◇ Sharpe Ratio $= \frac{E(r)-r_f}{\sigma}$

(2) Invest the entire budget in a risky asset for both years. Since the investment budget was subject to the risky asset for two years, the characteristics of the investment over the two-year horizon are as follows:

- ◇ Expected risk premium $= 2(E(r) - r_f)$
- ◇ SD $= \sqrt{2}\sigma$
- ◇ Sharpe Ratio $= \sqrt{2}\frac{E(r)-r_f}{\sigma}$
- ◇ Notice that this Sharpe Ratio is the same as the one we obtained above for risk pooling

(3) Invest half of the budget in the risky asset and half in the risk-free asset for both years. The characteristics of the investment over the two-year horizon are as follows:

- ◇ Expected risk premium = Expected risk premium (Year 1) + Expected risk premium (Year 2) $= \frac{1}{2}(E(r) - r_f) + \frac{1}{2}(E(r) - r_f) = (E(r) - r_f)$
- ◇ Variance = Variance (Year 1) + Variance (Year 2) $= (\frac{1}{2}\sigma)^2 + (\frac{1}{2}\sigma)^2 = \frac{1}{2}\sigma^2$
- ◇ SD $= \sqrt{\frac{1}{2}\sigma^2} = \sqrt{\frac{1}{2}}\sigma$

- ◇ Sharpe Ratio = $\sqrt{2}\frac{E(r)-r_f}{\sigma}$
- ◇ Notice that this Sharpe Ratio is the same as the one we obtained above for risk sharing

The bottom line is that it is preferable to invest in an efficient portfolio over many periods (the higher Sharpe Ratio shows this). However, in order to do so while reducing risk, an investor should also decrease the amount in the risky asset in each period.

Feldblum Financial

Outline

The objective of this paper is to demonstrate how an internal rate of return (IRR) model can be used to price insurance policies.

I. Introduction

Traditionally, insurance premiums included fixed underwriting profit margins (such as 5%) without theoretical justification. There are **three reasons for seeking more accurate pricing models**:

(1) **Time value of money** – Insurance cash flows occur at different times. For example, premiums/expenses are often collected/paid at policy inception, while losses are settled at a later date. Pricing models should seek to consider both the timing and magnitude of the cash flows

(2) **Competition and expected returns** – The price of a product depends on the degree of competition in the industry

(3) **Rate Base** – The traditional underwriting profit margins is a return on sales. A more appropriate rate base is a return on equity

Point of View

Insurance transactions can be examined from **two points of view**:

(1) Insurer ↔ Policyholder

- ◇ Policyholder pays premium for an insurance contract, which requires the insurer to indemnify the policyholder for covered claims
- ◇ Transactions occur in the product market
- ◇ Prices (i.e. premiums) are influenced by the supply of insurance coverage and demand for insurance services
- ◇ Profits are related to premiums and losses with no consideration of the insurer's capital structure
- ◇ Traditional rate-making procedures used this point of view

(2) Equity Provider $\leftrightarrow$ Insurer

- ⋄ Shareholders (i.e. equity providers) invest funds in an insurance company, expecting a return on that investment (i.e. capital accumulation or dividends) from the insurer
- ⋄ Transactions occur in the financial market
- ⋄ Expected returns driven by the risks of insurance operations
- ⋄ Profits are related to assets or equity with consideration of insurance cash flows (premiums, losses and expense) ONLY to the extent that they affect transactions between the insurer and its stockholder
- ⋄ IRR model uses this point of view

How are these **points of view related**?

- ⋄ The supply of insurance services in the product market depends on the costs that insurers pay to obtain capital, as well as the returns achievable by investors on other uses of that capital
- ⋄ The expected returns in the financial market, which are influenced by the risk of insurance operations, depend on consumers' demand for insurance services

A Non-Insurance Illustration of the IRR Model

The decision rule of the IRR model is as follows: **Accept an investment opportunity which offers a rate of return in excess of the opportunity cost of capital**. In other words, if the IRR is greater than the cost of capital, then the project is expected to be profitable and should be pursued.

As a simple **example**, consider a firm with $250,000 of annual revenue and $50,000 of annual expenses. For $100,000, the firm can purchase new equipment with a two year life span that would increase annual revenue to $300,000 and decrease annual expenses to $35,000. Should the firm purchase the new equipment?

Before we can answer this, we need assumptions around timing and taxes:

- ⋄ Purchase costs are incurred at the beginning of the year
- ⋄ Increases in revenue and decreases in expenses occur at the end of each year
- ⋄ No federal income taxes

We need to determine the IRR and compare it to the cost of capital. The IRR is the rate of return needed to set the net present value of cash flows to zero. In this case, we want to solve for R in the following equation:

$$0 = -100000 + \frac{65000}{1+R} + \frac{65000}{(1+R)^2}$$

where 65,000 = (300,000 - 250,000) + (50,000 - 35,000). Using the IRR function within the CBT environment, we find that $R = 19.4\%$. Assuming the cost of capital is less than 19.4% per annum, then the firm should purchase the equipment.

Quick Aside on the CBT IRR Function

Similar to Excel, the CBT environment has an IRR function. Here are the steps needed to solve this problem using the function:

(1) We are going to use two columns. To make it easy on the graders, let's label the columns "Time" and "Cash Flow"

(2) Under "Time," enter 0, 1 and 2 since the cash flows take place at times 0, 1 and 2. Note that the times are NOT inputs into the IRR function. Instead, the IRR function assumes that the cash flows provided occur at times 0, 1, 2, etc. However, we should label the time periods for clarity

(3) Under "Cash Flow," let's enter -100,000 in the cell next to Time 0, 65,000 in the cell next to Time 1 and 65,000 in the cell next to Time 2

(4) In a separate cell, enter the following IRR function: "=IRR(...)", where the "..." references the cash flows. Make sure to label this cell as IRR for clarity and ease of grading

The CBT environment should show 19.4%.

Insurance IRR Models

In the non-insurance example, there was an initial outflow (the purchase of the equipment) followed by future revenues (the $65,000 from additional revenue and reduced expenses). Property/liability insurance operations **appear to show the opposite pattern**: the insurer collects premiums before paying losses. However, this **ignores the equity commitments that support the insurance operations**.

Two aspects of insurance operations that reflect the equityholders' perspective are incorporated in IRR pricing models:

(1) When an insurer writes a policy, part of the premium is used to pay acquisition, underwriting and administrative expenses. The remaining premium dollars are invested to support the unearned premium reserve and the loss reserve

(2) Insurance companies commit surplus to ensure that that company has sufficient capital to withstand unexpected losses

The commitment of surplus underlies the equity flows of an insurer's owners. The owners must provide funds to allow the firms to write the policy. These funds are **cash outflows**. The return on these funds occurs in future years as the policy expires, losses are paid and the surplus is freed. These returns are cash inflows. Now we can see the similarities between the non-insurance IRR models and the insurance IRR models.

Equity Flows

Equity flows represent the cash outflows and cash inflows for the equityholders. These cash flows are impacted by insurance transactions (premiums, losses, expenses, etc.) in that higher levels of underwriting and investment income reduce the amount of surplus commitment needed from equityholders. *At this point, Feldblum goes into some detail on the level of surplus associated with certain lines of business. Since this is explored further later on in the paper, I will skip it for now and come back to it later.*

An Equity Flow Illustration

The following example is a modified version of the one shown in the paper. Suppose an insurer has an opportunity to write a new program with the following characteristics:

- ◇ Premium of $1,000 collected and earned at time 0
- ◇ Expected losses of $500 and $200 paid at time 1 and time 2, respectively
- ◇ Expenses are 30% of premium and are paid at time 0
- ◇ Loss reserves are established at time 0 and there are no unearned premium reserves
- ◇ Before-tax investment yield is 10%
- ◇ Tax rate on investment income is 35%
- ◇ Cost of capital is 8%
- ◇ The required reserve to surplus ratio 2:1

To determine the IRR for this model, it's simplest to create a table. I use the following table for every IRR problem:

	Time 0	Time 1	Time 2
Premium	1,000	0	0
Expense	300	0	0
Paid Loss	0	500	200
Loss Reserve	700	200	0
Required Surplus	350	100	0
Required Assets	1,050	300	0
Beginning Assets	1,000	1,118.25	319.50
Payments	300	500	200
Equity Flow (from Insurer's Perspective)	350	–318.25	–119.50
Equity Flow (from Equityholders' Perspective)	–350	318.25	119.50
Ending Assets	1,050	300	0

Here are the general formulas I used for the example above:

- ◇ Required Surplus $= \frac{\text{Loss Reserve}}{\text{Reserve to Surplus Ratio}}$
- ◇ Required Assets = Loss Reserves + Required Surplus
- ◇ Beginning Assets = Ending Assets × After-Tax Investment Yield (with Beginning Assets at Time 0 equal to Premium)
- ◇ Ending Assets = Required Assets
- ◇ Equity Flow (from Insurer's Perspective) = Required Assets + Payments – Beginning Assets
- ◇ Equity Flow (from Equityholders' Perspective) = –Equity Flow (from Insurer's Perspective)

To illustrate the formulas, let's look at time periods 0 and 1:

- ◇ Time 0
  - Expense = 1,000(0.30) = 300
  - Payments = 300
  - Required Surplus $= \frac{700}{2} = 350$
  - Beginning Assets = Premium = 1,000
  - Required Assets = Ending Assets = 700 + 350 = 1,050
  - Equity Flow (from Insurer's Perspective) = 1,050 + 300 – 1,000 = 350
  - Equity Flow (from Equityholders' Perspective) = –350
- ◇ Time 1
  - Payments = 500

- Loss Reserve = 700 - 500 = 200
- Required Surplus = $\frac{200}{2} = 100$
- Required Assets = Ending Assets = 200 + 100 = 300
- Beginning Assets = $1,050[1 + 0.10(1 - 0.35)] = 1,118.25$
- Equity Flow (from Insurer's Perspective) = 300 + 500 − 1,118.25 = –318.25
- Equity Flow (from Equityholders' Perspective) = 318.25

We want to solve for R in the following equation: $0 = -350 + \frac{318.25}{1+R} + \frac{119.50}{(1+R)^2}$. Notice that we are using the equity flows from the **equityholders' perspective**. Using the IRR function within the CBT environment, we find that $R = 19.5\%$. Since R is greater than the cost of capital ($19.5\% > 8\%$), the premiums are adequate from the equityholders' perspective and the program should be pursued

There is one thing that should be noted in the example above:

◇ Generally speaking, the UEPR is often assumed to be zero for these problems. The ultimate losses are assumed to be known at Time 0 and the initial surplus commitment is obtained by applying a reserve to surplus ratio to the loss reserves. However, it's possible for an exam problem to state the initial surplus requirement as a function of the UEPR. Make sure to look for this when working exam problems

II. Surplus

A insurance contract is a promise to compensate policyholders for future claims. An insurer must have financial assets to fulfill this promise. Unlike the fixed assets held by a manufacturer (ex. plants/equipment) which can be objectively measured, an insurer's surplus and its allocation to lines of business and time periods is theoretical. Thus, an IRR model must assume a relationship between surplus commitment and insurance transactions.

The Individual Firm and the Industry

The difference between insurance pricing and the uses of IRR models in other industries has several implications:

(1) In other industries, capital investments reflect a deliberate strategy. If a firm anticipates larger returns, it should invest more funds. In the insurance industry, capital markets are assumed to be efficient which means that the industry is neither over- nor under-capitalized. However, an individual insurer may be over- or under-capitalized due to the favorable or adverse operating results in the past. This implies that the actual surplus held by an insurer is not necessarily a deliberate strategy

(2) The IRR model focuses on the **capital/financial market** (return on equity vs. cost of capital) whereas the prices of insurance contracts are determined in the **product market** (supply and demand of insurance policies)

   - ◇ At the industry level, the capital and product markets are connected. If market prices are inadequate and industry returns are below the cost of capital, then firms will leave the industry and prices will rise
   - ◇ At the firm level, this relationship doesn't necessarily hold. An individual firm's IRR may be depressed by operating inefficiencies even when industry rates are adequate

Surplus Allocation

As mentioned earlier, the allocation of an insurer's surplus is theoretical. Surplus exists for the company as a whole and is available to all lines of business. However, an allocation of surplus to line of business is needed for the IRR model. There are two important questions related to this allocation:

(1) How is surplus allocated?

   - ◇ Base – Actuaries generally allocate surplus in proportion to a base, such as written premium, statutory reserves or the present value of future loss payments
   - ◇ Timing – Actuaries must understand when the surplus is committed and when the surplus is freed

(2) Should the surplus allocation depend on the type of policy?

   - ◇ For example, does a retrospectively rated Workers' Compensation policy require less surplus than a guaranteed cost policy?

Premiums and Reserves

If we use **premiums** as our surplus allocation base, it implies the following:

- ◇ Required surplus varies directly with premium. If one line has twice the premium of a second line, then it receives twice the surplus commitment
- ◇ Surplus is committed when the premium is written, and it is released when the policy expires

If we use **reserves** as our surplus allocation base, it implies the following:

- ◇ Required surplus varies directly with reserves. If one line has twice the reserves of a second line, then it receives twice the surplus commitment
- ◇ If the allocation base is loss and expense reserves only, then surplus is committed when the losses occur, and it is released when losses are paid

- ◇ If the allocation base includes the loss portion of the unearned premium reserves in addition to the loss reserves, then surplus is committed when the policy is written, and it is released when the losses are paid

When more surplus is allocated to a line of business, the internal rate of return moves closer to the company's after-tax investment yield. This is because an increase in surplus decreases the internal rate of return. Generally speaking, the insurance industry's cost of capital exceeds the after-tax investment yield so any decrease in the internal rate return will move it closer to the investment yield.

Long- and Short-Tailed Lines

Slow paying lines with large loss reserves (ex. Workers' Compensation, General Liability) receive a larger allocation of surplus if reserves are used as the base instead of premium. This is due to the period of time in which surplus is committed.

As an **example**, suppose an insurer with \$25 million in surplus writes two lines of business with the following characteristics over the next year:

	Line A	Line B
Written Premium	\$20 million	\$20 million
Loss Ratio	50%	40%
Average Time From Loss to Payment	3 years	1 years

Allocation of surplus by **premiums** gives the following:

- ◇ Line A Allocated Surplus = $\frac{20}{20+20}(25) = \$12.5$ million
- ◇ Line B Allocated Surplus = $\frac{20}{20+20}(25) = \$12.5$ million

Allocation of surplus by **reserves** gives the following:

- ◇ Line A Steady State Reserves = $20(0.50)(3) =\$30$ million
- ◇ Line B Steady State Reserves = $20(0.40)(1) =\$8$ million
- ◇ Line A Allocated Surplus = $\frac{30}{30+8}(25) = \$19.7$ million
- ◇ Line B Allocated Surplus = $\frac{8}{30+8}(25) = \$5.3$ million

Note the term "steady state reserves" above. By "steady state," we are referring to the reserves at any point in time in a steady state environment. For Line A, there are $20(0.50) =$ \$10 million in losses each year. Since the average time loss to payment is 3 years, there are \$30 million in reserves at any point in time.

If we had the underwriting income for each line of business over the next year, we could calculate the return on equity under each allocation basis above as ROE = $\frac{\text{UW Income}}{\text{Allocated Surplus}}$.

Insurance Risks

Surplus protects the insurer against the following risks:

- **Asset risk** – Risk that financial assets will depreciate (ex. bonds will default or stock prices will drop)
- **Pricing risk** – Risk that incurred losses and expenses will be greater than expected at policy expiration
- **Reserving risk** – Risk that loss reserves will not cover ultimate loss payments
- **Asset-liability Mismatch risk** – Risk that changes in interest rates will affect the market value of certain assets (ex. bonds) differently than that of liabilities
- **Catastrophe risk** – Risk that unforeseen losses (ex. hurricanes, earthquakes) will depress the return realized by the insurer
- **Reinsurance risk** – Risk that reinsurance recoverables will not be collected
- **Credit risk** – Risk that agents will not remit premium balances or that insureds will not remit accrued retrospective premiums

Some of the risks above (ex. pricing risk, catastrophe risk) occur during the policy period. The other risks continue until all losses are paid. For this reason, surplus is often allocated based on unpaid losses or a combination of unpaid losses and written premiums.

Policy Form

Risk varies by policy form:

- Occurrence contracts vs. Claims-made contracts
    - Occurrence contracts cover claims that occur during the policy term
    - Claims-made contracts cover claims that are reported during the policy term
    - Since they only cover reported claims, claims-made contracts eliminate much of the Incurred But Not Reported (IBNR) liability, which reduces the loss uncertainty. Thus, less surplus is needed to support these contracts
- Service contracts
    - Under service contracts, the insurer handles claims but does not incur loss liabilities. Thus, no insurance risk exists and no surplus is required for statutory financial statements

◇ Retrospective rating contracts

- Under retrospective rating, losses are fully reimbursed by the policyholder in the **primary layer**. There is "credit risk" present here due to the possibility that the insured fails to pay the premiums
- Outside of the primary layer (i.e. losses hit the loss limit or premiums hit the maximum premium), significant insurance risk to the insurer exists as they won't receive additional premium. This places the risk for these contracts between that of a service contract and a prospectively rated contract (such as occurrence and claims-made contracts)

◇ Other policy form examples include excess coverage and large deductible policies

As shown above, risk varies by policy form. Theoretically, we should allocate more surplus to riskier contracts. If we treat all policy forms equally, we would **overstate the risk** on retrospectively rated policies and **understate the risk on excess coverage**. If we only consider premiums and reserves, we would **understate the risk** on retrospectively rated, large deductible and excess policies. This is due to the fact that these three policy types cover claims that occur in higher loss layers where loss fluctuations are greater.

III. Potential Pitfalls in IRR Analyses

There are two common capital budgeting techniques that consider the time value of money:

(1) IRR – determines the interest rate that sets the present value of cash inflows to the present value of cash outflows. If this interest rate exceeds the cost of capital, then the project should be pursued. Otherwise, the project should be rejected. This is the technique we have been describing throughout the paper

(2) Net Present Value (NPV) – uses the **cost of capital** to discount all cash flows to the same time. If the sum of the discounted values is positive, then the project should be pursued. Otherwise, the project should be rejected.

As an **example** of the NPV technique, let's consider the equity flow illustration for an insurer that we looked at earlier. In that example, the equity flows **for the equityholder** were −350, 318.25 and 119.50 for times 0, 1 and 2, respectively. Rather than solving for the IRR, we must discount these flows to time 0 using the cost of capital of 8%. Thus, we find the following:

$$NPV = -350 + \frac{318.25}{1.08} + \frac{119.50}{(1.08)^2} = 47.13$$

Since the NPV is positive, we should pursue the program. In this case, the IRR and NPV techniques produced the same "accept or reject" decision. In fact, these two methods typically provide the same "accept or reject" decision. However, they may give different answers under the following scenarios:

- ◇ Budget constraints
- ◇ Mutual exclusive projects
- ◇ Unusual cash flows

A **criticism** of an IRR analysis is that it doesn't optimize the net worth of the corporation. Some argue that the NPV technique does this and should be used instead. Others counter this by stating that these arguments are not material from a practical standpoint.

Cash Flow Patterns

The cash flow patterns affect the number of positive solutions to the internal rate of return equation. The normal cash flow pattern is an initial cash outflow followed by a series of inflows. In this case, there is a single sign reversal as time progresses and one positive real root to the IRR equation. However, it's possible for projects to display an initial outflow followed by an inflow followed by another outflow. In this case, there are two sign reversals and "at most" two positive real roots to the IRR equation. If two positive real roots are produced, zero, one or both of them might be reasonable. To resolve this issue, analysts should seek to reframe the equation so that only one sign reversal (and hence, only one answer) occurs.

Oversimplifications

In general, multiple sign reversals in projected cash flows result from inaccuracies or oversimplifications. For example, suppose a specific cash flow item is spread evenly across four quarters as a simplification. If this is not realistic, this simplification may cause a sign reversal when combined with other cash flows that are allocated to each quarter more precisely.

Although sign reversals are often caused by oversimplifications, they can be legitimate as well. For example, an unexpected IBNR loss may cause a cash outflow from investors in the middle of the stream of inflows.

Mutually Exclusive Projects and Reinvestment Rates

When projects are mutually exclusive (ex. aggregate budget constraints or projects accomplish the same goal), NPV and IRR techniques may not give the same ranking of projects.

An an **example**, suppose an insurer is comparing the NPVs of two projects at various interest rates:

Project	Cash Flow at Time 0	Cash Flow at Time 1	Cash Flow at Time 2
A	–12,000	10,000	6,500
B	–12,000	5,000	12,500

Project	NPV at 10%	NPV at 15%	NPV at 20%	NPV at 25%	NPV at 30%
A	2,463	1,611	847	160	–462
B	2,876	1,800	847	0	–757

The NPV analysis shows that Project B is preferable to Project A at interest rates below 20%, whereas Project A is preferable to Project B at interest rates above 20%. Summarized another way, it is often wiser to defer income for a larger total return when interest rates are low (such as Project B). On the flip side, it is wiser to achieve income early when interest rates are high (such as Project A).

The IRR for Project A is 26.3% and the IRR for Project B is 25%. Based on the IRR rules, we should pursue Project A. This line of thinking leads to **another criticism** of the IRR analysis. Project A is only preferable **if the revenue received in the first year ($10,000) can be invested at 26.3%**. In reality, the firm's cost of capital is 15%, which represents the return that the firm's owners can receive on their funds. The IRR analysis **incorrectly mixes the interest rate that equates real values of inflows and outflows with the interest rate at which the firm can invest the funds it receives**.

Feldblum states that this point is not valid for the IRR insurance pricing model for the following reasons:

(1) The IRR pricing model is used to set statewide manual rates, not to price individual policies. If the cost of capital is 15%, but the pricing model shows an IRR of 20%, the insurer can reinvest the revenue it receives by writing more policies. Assuming the insurer can grow at the internal rate of return and maintain the same quality of risks, the IRR assumptions are correct

(2) When using the IRR pricing model to determine the UW profit provision, the actuary selects a premium rate that equalizes the IRR and the cost capital. This eliminates the difference between NPV and IRR analyses

Practical Criticisms

As discussed, an IRR analysis rejects a project if the IRR is less than the cost of capital (even if the IRR is positive and greater than the investment yield). This might confuse a regulator as a positive IRR appears to indicate profitability. Furthermore, an IRR above the investment yield seems to indicate that rates are more than sufficient.

From an intuition standpoint, the NPV method wins here. Since the NPV discounts cash flows at the cost of capital, it would show a negative number in the case where the IRR is below the cost of capital. A negative NPV clearly demonstrates that rates are not adequate. However, in an IRR pricing model, **as costs increase, the internal rate of return drops slowly**. This slow drop can create positive values even as premiums become severely inadequate.

This occurs because the required surplus in an IRR pricing model changes as costs change. For example, suppose required surplus is determined by a reserve to surplus ratio. Then, as costs increase, the surplus and investment income will also increase. Although the total return is inadequate, the added investment income offsets some of the UW loss, and the IRR declines more slowly with increases in loss.

This differs from a utility company where the equity base against which return are measured is a fixed amount (i.e. doesn't change as costs change).

Premium Inadequacies and IRR Analyses

In the case of rising costs but depressed premiums, the implied equity flow assumptions are not reasonable. In reality, equityholders will provide less capital or none at all. If properly interpreted, an IRR model will show this.

As an **example**, suppose the following is true for a policy:

- ◇ Premium of $10,000 is collected at policy inception
- ◇ One loss will be paid four years after policy inception
- ◇ The insurer funds the loss with a four year zero-coupon bond yielding 10% per year
- ◇ Required surplus assumes a 2:1 ratio of reserves to surplus
- ◇ Cost of capital is 15%

At an **expected loss of $12,000**, we obtain the following:

- ◇ At policy inception, total assets after equity contributions are $18,000. This is calculated as follows:
  - At policy inception, the UW loss is 10,000 - 12,000 = -2,000. Equityholders will fund the loss
  - Equityholders will also provide 6,000 of supporting surplus, where 6,000 = 12,000/2
  - Thus, at policy inception, the equity contributions are equal to 8,000 and total assets are equal to 10,000 + 8,000 = 18,000
- ◇ The total assets grow to $26,354 after four years in a zero-coupon bond
- ◇ At time 4, the loss of $12,000 is paid and $14,354 is return to the equityholders
- ◇ The IRR found by solving the following equation: $0 = -8000 + \frac{14354}{(1+R)^4}$. Thus, R = IRR = 15.74%. Since this is greater than the cost of capital of 15%, this policy is acceptable

At an **expected loss of $15,000**, we obtain the following:

- ◇ Before considering the IRR analysis, we can see that this contract is unprofitable. The premium of $10,000 accumulates to $14,641 after four years at 10%. This means that the premium plus investment income is not enough to cover the loss of $15,000. Part of the loss must be funded with existing surplus
- ◇ An IRR analysis will lead to the same conclusion if interpreted properly
- ◇ At policy inception, total assets after equity contributions are $22,500 (10,000 + 12,500). These grow to $32,942 after four years in a zero-coupon bond
- ◇ At time 4, The loss of $15,000 is paid and $17,942 is return to the equityholders
- ◇ The IRR found by solving the following equation: $0 = -12500 + \frac{17942}{(1+R)^4}$. Thus, R = IRR = 9.46%. Since this is less than the cost of capital of 15%, this policy is not acceptable. Furthermore, since the IRR is less than the investment yield, the operating ratio exceeds 100%, and existing surplus must be used to fund the loss

We must increase the expected losses considerably before a negative IRR occurs. But it's clear that we have profitability issues well before that.

At the end of the day, the problem boils down to rate filing presentation and acceptance by regulators. When the contract is unprofitable, Feldblum suggests that the actuary show the negative expected NPV (from the NPV technique) and the insufficiency of net premiums plus investment income to fund the losses. However, the IRR analysis can still be used internally.

