Rising Fellow

2026

Exam 7

Study Guide

ADVANCED ESTIMATION OF CLAIMS LIABILITIES

Comprehensive study guide
with original and past CAS problems

Exam 7 Study Guide

Spring 2026 Sitting

Rising Fellow

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Introduction

How to Use This Guide

This guide is intended to **supplement** the Content Outline readings. Although we believe it provides a thorough review of the exam material, the readings provide additional context that is invaluable. Please do NOT skip the Content Outline readings.

Past CAS Exam Problems

Past CAS exam problems & solutions are included for each paper. Note that these questions are solely owned by the CAS. They are included in the online course for student convenience. All past CAS problems are Excel-based and can be downloaded from the online course.

Rounding

Numerous examples are provided throughout the guide. We did ***not round*** any of the intermediate steps in the examples. All calculations were performed with full precision to ensure accuracy, and only the final answers were rounded when necessary. To exactly reproduce the examples, we recommend working them out in Microsoft Excel.

Feedback

We always working to improve the Exam 7 Study Guide and the rest of the Rising Fellow study material. Please send us an email at **exam7@risingfellow.com** if you have feedback about any of the following:

- Sections that are confusing or could be improved
- Errors (ex. formatting, spelling, calculations, grammar, etc.)

Note that errata will be posted on the Rising Fellow website on an as-needed basis.

Blank Pages

Since many students want a printed copy of the study guide, blank pages have been inserted throughout the guide to ensure that all outlines start on odd pages.

Mack – Benktander

Outline

This paper focuses on the Benktander method and its relationship to the chain-ladder & Bornhuetter/Ferguson methods.

I. General Relationship Between Reserve & Ultimate Loss Estimates

Suppose that C_k is the actual claims amount paid after k years of development. Given a **reserve estimate** $\hat{R}$ and **ultimate loss estimate** $\boldsymbol{\hat{U}}$, we have the following **general relationship**:

$$\boxed{\hat{U} = C_k + \hat{R}}$$

This relationship always holds. If we have a reserve estimate and add the claims paid to date, we always obtain the corresponding ultimate loss estimate.

II. Bornhuetter/Ferguson (BF) Method

The **BF method estimates reserves** based on an **a priori** expectation of ultimate losses. Mathematically:

$$\boxed{R_{BF} = q_k U_0}$$

where:

- R_{BF} is the BF reserve
- $q_k = 1 - \frac{1}{CDF}$ is the proportion of the ultimate claims amount which is expected to remain unpaid after k years of development
- U_0 is the a priori expectation of ultimate losses

Since R_{BF} uses U_0, it assumes the current claims amount C_k is **NOT predictive of future claims**.

Using the general relationship described above, we obtain the **BF ultimate loss**:

$$\boxed{U_{BF} = C_k + R_{BF}}$$

III. Chain-Ladder (CL) Method

The **CL method** estimates ultimate losses and reserves based on claims to date. In other words, it assumes that the current claims amount C_k is **fully predictive of future claims**. Mathematically:

$$\boxed{U_{CL} = \frac{C_k}{p_k}} \qquad \boxed{R_{CL} = q_k U_{CL}}$$

where:

- U_{CL} is the **chain-ladder ultimate loss**
- $p_k = 1 - q_k$ is the proportion of the ultimate claims amount which is expected to be paid after k years of development

The **main advantage of the CL method** over the BF method is different actuaries should obtain similar results when running the chain-ladder method. This is not the case with the BF method due to differences in the selection of U_0.

IV. Benktander Method

Since the CL and BF methods represent extreme positions, where the CL method fully believes C_k and the BF method does not rely on C_k at all, Gunnar Benktander replaced U_0 with a credibility mixture:

$$U_c = cU_{CL} + (1 - c)U_0$$

where c is the credibility weight.

As the claims C_k develop, credibility should increase, Benktander proposed the following:

- Set $c = p_k$

- Set $R_{GB} = R_{BF} \cdot \frac{U_{p_k}}{U_0} = q_k U_0 \cdot \frac{U_{p_k}}{U_0} = q_k U_{BF}$, where R_{GB} is the Benktander reserve

Benktander's proposals lead to some important results.

BF Method as a Credibility-Weighted Average

Using our credibility mixture with $c = p_k$, we can show the following:

$$\begin{aligned} U_{p_k} &= p_k U_{CL} + (1 - p_k) U_0 \\ &= p_k U_{CL} + q_k U_0 \\ &= C_k + R_{BF} \\ &= U_{BF} \end{aligned}$$

Hence, the **BF method** is a credibility-weighted average of the CL method and the a priori expectation.

Benktander Method as a Credibility-Weighted Average

The **Benktander method** is a credibility-weighted average of the CL and BF methods:

$$\boxed{\begin{aligned} U_{GB} &= C_k + R_{GB} \\ &= p_k U_{CL} + q_k U_{BF} \\ &= (1 - q_k) U_{CL} + q_k U_{BF} \end{aligned}}$$

The **Benktander reserve** is also a credibility-weighted average of the CL and BF methods:

$$\boxed{R_{GB} = (1 - q_k) R_{CL} + q_k R_{BF}}$$

We **can also express the Benktander** method as a credibility-weighted average of the CL method and the a priori expectation:

$$\begin{aligned} U_{GB} &= C_k + R_{GB} \\ &= (1 - q_k^2) U_{CL} + q_k^2 U_0 \end{aligned}$$

Iterative Relationships

As shown above, the Benktander reserve is obtained by applying the BF method twice, first to U_0 (which produces R_{BF}), and then to U_{BF} (which produces R_{GB}). Hence, the Benktander method is also called the **iterated BF method**.

Using a starting point of $U^{(0)} = U_0$ and iteration rules $R^{(m)} = q_k U^{(m)}$ and $U^{(m+1)} = C_k + R^{(m)}$, we obtain the following **iterative relationships**:

$$U^{(m)} = (1 - q_k^m)U_{CL} + q_k^m U_0$$
$$R^{(m)} = (1 - q_k^m)R_{CL} + q_k^m R_{BF}$$

These relationships lead to the following:

- $U^{(0)} = (1 - q_k^0)U_{CL} + q_k^0 U_0 = U_0$
- $R^{(0)} = (1 - q_k^0)R_{CL} + q_k^0 R_{BF} = R_{BF}$
- $U^{(1)} = (1 - q_k)U_{CL} + q_k U_0 = C_k + R_{BF} = U_{BF}$ ← Iteration 1 Ultimate
- $R^{(1)} = (1 - q_k)R_{CL} + q_k R_{BF} = R_{GB}$ ← Iteration 2 Reserve
- $U^{(2)} = (1 - q_k^2)U_{CL} + q_k^2 U_0 = U_{GB}$ ← Iteration 2 Ultimate
- …
- $U^{(\infty)} = (1 - q_k^\infty)U_{CL} + q_k^\infty U_0 = U_{CL}$
- $R^{(\infty)} = (1 - q_k^\infty)R_{CL} + q_k^\infty R_{BF} = R_{CL}$

As shown above, the BF and Benktander methods are iterations 1 and 2, respectively. If we **infinitely iterate between reserves and ultimate losses**, we will eventually obtain the **CL method**.

V. Benktander vs. BF vs. CL

The Benktander method is superior to the BF and CL methods for the following reasons:

- The mean squared error (MSE) is almost always smaller than the BF or CL methods
- Better approximation of the exact Bayesian procedure

- Superior to the CL method since it gives more weight to the a priori expectation of ultimate losses
- Superior to the BF method since it gives more weight to actual loss experience

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Example: Benktander Method

An actuary wants to use the Benktander method to estimate ultimate losses for accident year 2021. Given the following information as of December 31, 2023:

	Cumulative Paid Losses ($) as of XX Months:			
AY	12 mo.	24 mo.	36 mo.	48 mo.
2020	7,000	10,500	12,600	13,860
2021	8,000	12,000	14,400	
2022	9,000	13,500		
2023	10,000			

- The 2021 earned premium is $25,000
- The expected loss ratio for each year is 75%
- The 48-ultimate loss development factor is 1.05

First, let's calculate the BF ultimate losses:

- Since AY 2021 is 36 months old (i.e., $k = 3$), we need the 36-ultimate CDF
  - The 36-48 LDF is $\frac{13{,}860}{12{,}600} = 1.10$
  - The 36-ultimate CDF is 1.10(1.05) = 1.155
- $U_0 = EP \cdot ELR = 25{,}000(0.75) = 18{,}750$
- $R_{BF} = q_3 U_0 = \left(1 - \frac{1}{1.155}\right)(18{,}750) = 2{,}516$
- $U_{BF} = C_3 + R_{BF} = 14{,}400 + 2{,}516 = 16{,}916$

Second, let's calculate the Benktander ultimate losses:

- $R_{GB} = q_3 U_{BF} = \left(1 - \frac{1}{1.155}\right)(16{,}916) = 2{,}270$
- $U_{GB} = C_3 + R_{GB} = 14{,}400 + 2{,}270 = 16{,}670$

The Benktander ultimate losses for AY 2021 are **$16,670**.

Note that there are a number of other ways to calculate this quantity. As long as you reproduce the same figure using any of the applicable formulas, you should be good to go.

Hürlimann

Outline

The methods shown in this paper are inspired by those shown in the Mack Benktander paper. The main difference is that this paper focuses on loss ratios rather than raw losses.

I. The Collective & Individual Loss Ratio Claims Reserves

This section of the paper defines a number of quantities and builds out the collective and individual loss ratio claims reserves. When first reviewing the formulas below, don't stress about the notation. Once you look at an example and start working problems, it will all come together.

Total Ultimate Claims

Let i represent the origin period (typically an accident year) and n be the total number of origin periods. Assuming that all claims incurred in an origin period are known and closed after n development periods, the **total ultimate claims**, U_i, from origin period i are as follows:

$$U_i = \sum_{k=1}^{n} S_{ik}$$

where S_{ik} are the *incremental paid claims* from origin period i.

Cumulative Paid Claims

Cumulative paid claims from origin period i as of k years of development, C_{ik}, are defined as follows:

$$C_{ik} = \sum_{j=1}^{k} S_{ij}$$

i-th Period Claims Reserve

The i-th period claims reserve, R_i, is the required amount for the incurred but unpaid claims of period i. It is defined as follows:

$$R_i = \sum_{k=n-i+2}^{n} S_{ik}$$

where $i = 2, \ldots, n$. This means that we assume that the oldest origin period is already at ultimate.

Total Claims Reserve

The total amount of incurred but unpaid claims over all periods, R, is defined as follows:

$$R = \sum_{i=2}^{n} R_i$$

Incremental Expected Loss Ratio

Let V_i be the **premium** for origin period $\boldsymbol{i}$. The incremental amount of expected paid claims per unit of premium in each development period, m_k, is defined as follows:

$$m_k = \frac{E\left[\sum_{i=1}^{n-k+1} S_{ik}\right]}{\sum_{i=1}^{n-k+1} V_i}$$

Based on the definition above, m_k is the incremental expected loss ratio for each development period. If we *sum* m_k across development periods, we obtain the **total expected loss ratio**.

Expected Value of the Burning Cost

The expected value of the burning cost of the total ultimate claims is similar to the a priori estimate U_0 from the Mack Benktander paper. It is defined as follows:

$$E\left[U_i^{BC}\right] = V_i \cdot \sum_{k=1}^{n} m_k$$

The formula above is multiplying the premium for origin period i by the total expected loss ratio to arrive at the expected value of the burning cost (i.e., expected loss).

Loss Ratio Payout & Reserve Factors

The **loss ratio payout factor**, p_i, represents the percentage of losses paid to date for each origin period. It is defined as follows:

$$p_i = \frac{V_i \cdot \sum_{k=1}^{n-i+1} m_k}{E[U_i^{BC}]} = \frac{\sum_{k=1}^{n-i+1} m_k}{\sum_{k=1}^{n} m_k}$$

The **loss ratio reserve factor**, q_i, represents the percentage of losses yet to be paid for each origin period. It is defined as follows:

$$q_i = 1 - p_i$$

These quantities are similar to p_k and q_k from the Mack Benktander paper.

Individual Total Ultimate Claims & Loss Ratio Claims Reserve

The **individual total ultimate claims** for origin period i, U_i^{ind}, are defined as follows:

$$U_i^{ind} = \frac{C_{i,n-i+1}}{p_i}$$

The **individual loss ratio claims reserve** for origin period i, R_i^{ind}, is defined as follows:

$$\begin{aligned} R_i^{ind} &= U_i^{ind} - C_{i,n-i+1} \\ &= q_i \cdot U_i^{ind} \\ &= \frac{q_i}{p_i} \cdot C_{i,n-i+1} \end{aligned}$$

These quantities are similar to the CL ultimate loss & reserve from the Mack Benktander paper.

Collective Total Ultimate Claims & Loss Ratio Claims Reserve

The **collective loss ratio claims reserve** for origin period i, R_i^{coll}, is defined as follows:

$$R_i^{coll} = q_i \cdot U_i^{BC}$$

The **collective total ultimate claims** for origin period i, U_i^{coll}, is defined as follows:

$$U_i^{coll} = R_i^{coll} + C_{i,n-i+1}$$

These quantities are similar to the BF ultimate loss & reserve from the Mack Benktander paper.

An **advantage** of the collective loss ratio claims reserve over the standard BF reserve is that different actuaries always come to the same results provided they use the same premiums. This is because the expected loss ratio is based on the actual claims data rather than a judgmental selection.

II. Credible Loss Ratio Claims Reserve

Similar to the CL and BF reserves, the individual and collective loss ratio claims reserves represent extreme positions (fully believe the data vs. do not believe the data at all). Thus, it's sensible to define a **credibility-weighted** average of the individual and collective loss ratio reserves as follows:

$$R_i^c = Z_i \cdot R_i^{ind} + (1 - Z_i) \cdot R_i^{coll}$$

where Z_i is the credibility weight.

Benktander (GB) Loss Ratio Claims Reserve

If we set $Z_i = Z_i^{GB} = p_i$, we obtain the Benktander loss ratio claims reserve, R_i^{GB}:

$$\boxed{R_i^{GB} = p_i \cdot R_i^{ind} + q_i \cdot R_i^{coll}}$$

Neuhaus (WN) Loss Ratio Claims Reserve

If we set $Z_i = Z_i^{WN} = \sum_{k=1}^{n-i+1} m_k = p_i \cdot \sum_{k=1}^{n} m_k$, we obtain the Neuhaus loss ratio claims reserve, R_i^{WN}:

$$\boxed{R_i^{WN} = Z_i^{WN} \cdot R_i^{ind} + (1 - Z_i^{WN}) \cdot R_i^{coll}}$$

Iterative Relationships

Using a starting point of ${U_i}^{(0)} = U_i^0 = U_i^{BC}$ and iteration rules ${R_i}^{(m)} = q_i \cdot {U_i}^{(m)}$ and ${U_i}^{(m+1)} = C_{i,n-i+1} + {R_i}^{(m)}$, we obtain the following **iterative relationships**:

$$\boxed{\begin{aligned} {U_i}^{(m)} &= (1 - q_i^m) \cdot U_i^{ind} + q_i^m \cdot U_i^{BC} \\ {R_i}^{(m)} &= (1 - q_i^m) \cdot R_i^{ind} + q_i^m \cdot R_i^{coll} \end{aligned}}$$

Note that in the paper, the reserve relationship is as defined as ${R_i}^{(m)} = (1 - q_i^m) \cdot R_i^{ind} + q_i^m \cdot R_i^0$. Since $R_i^{coll} = q_i \cdot U_i^{BC}$, it must be the case that $R_i^{coll} = R_i^0$. Thus, these iteration rules are similar to those shown in the Mack Benktander paper.

These relationships lead to the following:

- $U_i^{(0)} = (1 - q_i^0) \cdot U_i^{ind} + q_i^0 \cdot U_i^{BC} = U_i^{BC}$
- $R_i^{(0)} = (1 - q_i^0) \cdot R_i^{ind} + q_i^0 \cdot R_i^{coll} = R_i^{coll}$
- $U_i^{(1)} = (1 - q_i) \cdot U_i^{ind} + q_i \cdot U_i^{BC} = U_i^{coll}$ ⟵ Iteration 1 Ultimate
- $R_i^{(1)} = (1 - q_i) \cdot R_i^{ind} + q_i \cdot R_i^{coll} = R_i^{GB}$ ⟵ Iteration 2 Reserve

- $U_i^{(2)} = (1 - q_i^2) \cdot U_i^{ind} + q_i^2 \cdot U_i^{BC} = U_i^{GB}$ ⟵ Iteration 2 Ultimate
- …
- $U_i^{(\infty)} = (1 - q_i^\infty) \cdot U_i^{ind} + q_i^\infty \cdot U_i^{BC} = U_i^{ind}$
- $R_i^{(\infty)} = (1 - q_i^\infty) \cdot R_i^{ind} + q_i^\infty \cdot R_i^{coll} = R_i^{ind}$

As shown above, the collective and Benktander methods are iterations 1 and 2, respectively. The labeling is a bit strange because $R_i^{(1)}$ is the Benktander reserve and uses a "1" instead of a "2." But the CAS made it clear that the Benktander method is considered the second-iteration reserve in the official solution for 2016 #1.

Once again, if we **infinitely iterate between reserves and ultimate losses**, we will eventually obtain the **individual** method.

III. The Optimal Credibility Weights & the Mean Squared Error

In this section, we summarize the ***key formulas*** from Sections 4 – 6 of the paper regarding the optimal credibility weights.

General Formula for the Optimal Credibility Weights

Assume that $Var(U_i) = f_i \cdot Var\left(U_i^{BC}\right)$ for some constant $f_i \geq 1$. This implies that U_i is *at least as volatile* as the burning cost estimate.

The optimal credibility weights Z_i^* which minimize the mean squared error $mse(R_i^c) = E[(R_i^c - R_i)^2]$ and the variance $Var(R_i^c)$ are as follows:

$$\boxed{Z_i^* = \frac{p_i}{p_i + t_i^*}}$$

where $t_i^* = \frac{f_i - 1 + \sqrt{(f_i + 1) \cdot (f_i - 1 + 2 \cdot p_i)}}{2}$.

Special Case of the Optimal Credibility Weights

The **minimum variance optimal credible claims reserve** is obtained when $f_i = 1$. In other words, of all possible values for f_i, $f_i = 1$ yields the lowest possible reserve variance. In this special case, $Var(U_i) = Var\left(U_i^{BC}\right)$ and $t_i^* = \sqrt{p_i}$. Then, the optimal credibility weights are as follows:

$$\boxed{Z_i^* = \frac{p_i}{p_i + \sqrt{p_i}}}$$

Since $t_i^* = \sqrt{p_i} \leq 1$, it must the case that $Z_i^* \leq \frac{1}{2}$. This clearly demonstrates that the Neuhaus and Benktander methods are not optimal since they often have weights greater than 0.5.

On the exam, the default formula for the optimal credibility weights is $Z_i^* = \frac{p_i}{p_i+\sqrt{p_i}}$. This applies in the following situations:

- $f_i = 1$
- $Var(U_i) = Var\left(U_i^{BC}\right)$

If you are told that $f_i > 1$ or that $Var(U_i) = f_i \cdot Var\left(U_i^{BC}\right)$, then use the general formula.

Mean Squared Error Formula

The mean squared error for the credible loss ratio claims reserve is as follows:

$$\boxed{mse(R_i^c) = E[\alpha_i^2(U_i)] \cdot \left(\frac{Z_i^2}{p_i} + \frac{1}{q_i} + \frac{(1 - Z_i)^2}{t_i^*}\right) \cdot q_i^2}$$

Notice that we have a new term, $E[\alpha_i^2(U_i)]$. This comes from the beginning of Section 4 where Hürlimann lays out the theory for the optimal credibility weights. For the purposes of the *exam*, just know that it's a constant that must be incorporated into the MSE calculation. It has only shown up one time on released exams (see Spring 2018 #3).

The formula above can also be used to obtain the MSEs of the **collective and individual methods**:

- When $Z_i = 0$, we obtain $mse\left(R_i^{coll}\right)$
- When $Z_i = 1$, we obtain $mse\left(R_i^{ind}\right)$

Example: Credible Loss Ratio Claims Reserves

Given the following information as of December 31, 2023:

AY	EP ($)	12 mo.	24 mo.	36 mo.	48 mo.	60 mo.	72 mo.
		Incremental Paid Losses ($) as of XX Months:					
2018	13,085	4,370	1,923	3,999	2,168	1,200	647
2019	14,258	2,701	2,590	1,871	1,783	393	
2020	16,114	4,483	2,246	3,345	1,068		
2021	15,142	3,254	2,550	2,547			
2022	16,905	8,010	4,108				
2023	20,224	5,582					

Assuming that $Var(U_i) = Var\left(U_i^{BC}\right)$, let's calculate the various reserves covered in the paper.

First, we need the applicable parameters:

- $m_k = \frac{E\left[\sum_{i=1}^{n-k+1} S_{ik}\right]}{\sum_{i=1}^{n-k+1} V_i}$
  - $m_1 = \frac{4{,}370+2{,}701+\cdots+5{,}582}{13{,}085+14{,}258+\cdots+20{,}224} = 0.2967$
  - $m_2 = \frac{1{,}923+2{,}590+\cdots+4{,}108}{13{,}085+14{,}258+\cdots+16{,}905} = 0.1777$
  - …
  - $m_6 = \frac{647}{13{,}085} = 0.0494$
- $p_i = \frac{\sum_{k=1}^{n-i+1} m_k}{\sum_{k=1}^{n} m_k} = Z_i^{GB}$
  - $p_1 = \frac{0.2967+0.1777+\cdots+0.0494}{0.2967+0.1777+\cdots+0.0494} = 1.0000$
  - $p_2 = \frac{0.2967+0.1777+\cdots+0.0583}{0.2967+0.1777+\cdots+0.0494} = 0.9450$
  - …

- $p_6 = \frac{0.2967}{0.2967+0.1777+\cdots+0.0494} = 0.3303$

- $q_i = 1 - p_i$
  - $q_1 = 1 - p_1 = 1 - 1 = 0.0000$
  - $q_2 = 1 - p_2 = 1 - 0.9450 = 0.0550$
  - …
  - $q_6 = 1 - p_6 = 1 - 0.3303 = 0.6697$
- $Z_i^{WN} = \sum_{k=1}^{n-i+1} m_k$
  - $Z_1^{WN} = 0.2967 + 0.1777 + \cdots + 0.0494 = 0.8983$
  - $Z_2^{WN} = 0.2967 + 0.1777 + \cdots + 0.0583 = 0.8488$
  - …
  - $Z_6^{WN} = 0.2967$
- $Z_i^* = \frac{p_i}{p_i+\sqrt{p_i}}$
  - $Z_1^* = \frac{p_1}{p_1+\sqrt{p_1}} = \frac{1.0000}{1.0000+\sqrt{1.0000}} = 0.5000$
  - $Z_2^* = \frac{p_2}{p_2+\sqrt{p_2}} = \frac{0.9450}{0.9450+\sqrt{0.9450}} = 0.4929$
  - …
  - $Z_6^* = \frac{p_6}{p_6+\sqrt{p_6}} = \frac{0.3303}{0.3303+\sqrt{0.3303}} = 0.3650$

The applicable parameters are summarized as follows:

			Parameters		
AY	m_k	$p_i = Z_i^{GB}$	q_i	Z_i^{WN}	Z_i^*
2018 ($i = 1$)	0.2967	1.0000	0.0000	0.8983	0.5000
2019 ($i = 2$)	0.1777	0.9450	0.0550	0.8488	0.4929
2020 ($i = 3$)	0.2007	0.8801	0.1199	0.7906	0.4840
2021 ($i = 4$)	0.1155	0.7515	0.2485	0.6751	0.4644
2022 ($i = 5$)	0.0583	0.5281	0.4719	0.4744	0.4209
2023 ($i = 6$)	0.0494	0.3303	0.6697	0.2967	0.3650

Second, let's calculate the collective, individual, Neuhaus, Benktander, and optimal claims reserves:

	Reserves ($)				
AY	Collective	Individual	Neuhaus	Benktander	Optimal
2019	705	544	568	553	626
2020	1,736	1,518	1,564	1,544	1,630
2021	3,380	2,761	2,962	2,915	3,092
2022	7,166	10,829	8,904	9,101	8,708
2023	12,167	11,320	11,916	11,887	11,858
Total	25,154	26,972	25,913	25,999	25,914

Notice that there are no reserves for AY 2018 since we assume that the oldest origin period is fully developed.

The calculations for AY 2019 ($i = 2$) are as follows:

- $R_2^{coll} = q_2 \cdot U_2^{BC} = q_2 \cdot V_2 \cdot \sum_{k=1}^{n} m_k = 0.0550(14{,}258)(0.2967 + 0.1777 + \cdots + 0.0494) = 705$
- $R_2^{ind} = \frac{q_2}{p_2} \cdot C_{2,5} = \frac{0.0550}{0.9450}(2{,}701 + 2{,}590 + \cdots + 393) = 544$
- $R_2^{WN} = Z_2^{WN} \cdot R_2^{ind} + (1 - Z_2^{WN}) \cdot R_2^{coll} = 0.8488(544) + (1 - 0.8488)(705) = 568$
- $R_2^{GB} = Z_2^{GB} \cdot R_2^{ind} + (1 - Z_2^{GB}) \cdot R_2^{coll} = 0.9450(544) + (1 - 0.9450)(705) = 553$
- $R_2^{opt} = Z_2^{*} \cdot R_2^{ind} + (1 - Z_2^{*}) \cdot R_2^{coll} = 0.4929(544) + (1 - 0.4929)(705) = 626$

Now that we have the reserves by AY for each method, we can easily calculate the ultimate losses by AY for each method by adding the AY paid claims to date to each reserve figure.

Third, let's calculate the MSEs for each method relative to the MSE of the optimal reserve:

	Relative MSE				
AY	Collective	Individual	Neuhaus	Benktander	Optimal
2019	1.0271	1.0287	1.0141	1.0228	1.0000
2020	1.0580	1.0659	1.0233	1.0389	1.0000
2021	1.1154	1.1535	1.0238	1.0441	1.0000

2022	1.1986	1.3761	1.0032	1.0129	1.0000
2023	1.2444	1.7401	1.0086	1.0022	1.0000

Recall that $mse(R_i^c) = E[\alpha_i^2(U_i)] \cdot \left(\frac{Z_i^2}{p_i} + \frac{1}{q_i} + \frac{(1-Z_i)^2}{t_i^*}\right) \cdot q_i^2$. The calculations for AY 2019 ($i = 2$) are as follows:

- Collective Relative MSE = $\frac{E[\alpha_2^2(U_2)]\cdot\left(\frac{0^2}{0.9450}+\frac{1}{0.0550}+\frac{(1-0)^2}{\sqrt{0.9450}}\right)\cdot 0.0550^2}{E[\alpha_2^2(U_2)]\cdot\left(\frac{0.4929^2}{0.9450}+\frac{1}{0.0550}+\frac{(1-0.4929)^2}{\sqrt{0.9450}}\right)\cdot 0.0550^2} = 1.0271$
- Individual Relative MSE = $\frac{E[\alpha_2^2(U_2)]\cdot\left(\frac{1^2}{0.9450}+\frac{1}{0.0550}+\frac{(1-1)^2}{\sqrt{0.9450}}\right)\cdot 0.0550^2}{E[\alpha_2^2(U_2)]\cdot\left(\frac{0.4929^2}{0.9450}+\frac{1}{0.0550}+\frac{(1-0.4929)^2}{\sqrt{0.9450}}\right)\cdot 0.0550^2} = 1.0287$
- Neuhaus Relative MSE = $\frac{E[\alpha_2^2(U_2)]\cdot\left(\frac{0.8488^2}{0.9450}+\frac{1}{0.0550}+\frac{(1-0.8488)^2}{\sqrt{0.9450}}\right)\cdot 0.0550^2}{E[\alpha_2^2(U_2)]\cdot\left(\frac{0.4929^2}{0.9450}+\frac{1}{0.0550}+\frac{(1-0.4929)^2}{\sqrt{0.9450}}\right)\cdot 0.0550^2} = 1.0141$
- Benktander Relative MSE = $\frac{E[\alpha_2^2(U_2)]\cdot\left(\frac{0.9450^2}{0.9450}+\frac{1}{0.0550}+\frac{(1-0.9450)^2}{\sqrt{0.9450}}\right)\cdot 0.0550^2}{E[\alpha_2^2(U_2)]\cdot\left(\frac{0.4929^2}{0.9450}+\frac{1}{0.0550}+\frac{(1-0.4929)^2}{\sqrt{0.9450}}\right)\cdot 0.0550^2} = 1.0228$
- Optimal Relative MSE = $\frac{E[\alpha_2^2(U_2)]\cdot\left(\frac{0.4929^2}{0.9450}+\frac{1}{0.0550}+\frac{(1-0.4929)^2}{\sqrt{0.9450}}\right)\cdot 0.0550^2}{E[\alpha_2^2(U_2)]\cdot\left(\frac{0.4929^2}{0.9450}+\frac{1}{0.0550}+\frac{(1-0.4929)^2}{\sqrt{0.9450}}\right)\cdot 0.0550^2} = 1.0000$
- Notice that the only quantities being swapped out for each calculation are the credibility weights in the numerator. In addition, notice that we don't need $E[\alpha_2^2(U_2)]$ since it cancels out

Using the relative MSE table above, it's clear that the Neuhaus reserve best matches the optimal reserve since the relative MSEs are the closest to 1.0000.

IV. Reinterpreting Traditional Reserving Methods

At the end of Section 6, Hürlimann reinterprets traditional reserving methods using the approaches shown in this paper.

Chain-Ladder Method

The chain-ladder method is similar to the individual loss ratio method with the loss ratio payout factors replaced with the standard chain-ladder lag factors, defined as $p_i^{CL} = \frac{1}{CDF}$, where the CDF is based on volume-weighted development factors.

Hence, the chain-ladder reserve is as follows:

$$R_i^{CL} = \frac{q_i^{CL}}{p_i^{CL}} \cdot C_{i,n-i+1}$$

Cape Cod Method

Hürlimann defines the Cape Cod method as a Benktander-type credibility mixture with the following components:

- $R_i^{ind} = \frac{q_i^{CL}}{p_i^{CL}} \cdot C_{i,n-i+1}$
- $R_i^{coll} = q_i^{CL} \cdot ELR \cdot V_i$, where $ELR = \frac{\sum_{i=1}^{n} C_{i,n-i+1}}{\sum_{i=1}^{n} p_i^{CL} \cdot V_i}$
- $Z_i = p_i^{CL}$

There are a couple of things to note in the formulas above:

1) The loss ratio payout factors have been replaced with the standard chain-ladder lag factors
2) The definition of R_i^{coll} above is the standard Cape Cod reserve. So technically, Hürlimann's "Cape Cod" method is a credibility-weighted average of the standard chain-ladder and Cape Cod methods

Optimal Cape Cod Method

The optimal Cape Cod method is identical to the Cape Cod method above, but with the following credibility weights:

$$Z_i = \frac{p_i^{CL}}{p_i^{CL} + \sqrt{p_i^{CL}}}$$

<u>BF Method</u>

Hürlimann defines the BF method as a Benktander-type credibility mixture with the following components:

- $R_i^{ind} = \frac{q_i^{CL}}{p_i^{CL}} \cdot C_{i,n-i+1}$
- $R_i^{coll} = q_i^{CL} \cdot ELR_i \cdot V_i$, where ELR_i is some selected initial loss ratio for each origin period
- $Z_i = p_i^{CL}$

Similar to the Cape Cod method, Hürlimann's "BF" method is a credibility-weighted average of the standard chain-ladder and BF methods.

<u>Optimal BF Method</u>

The optimal BF method is identical to the BF method above, but with the following credibility weights:

$$Z_i = \frac{p_i^{CL}}{p_i^{CL} + \sqrt{p_i^{CL}}}$$

Brosius

Outline

I. Introduction

Real-world loss data is subject to the following:

- Random fluctuations
- Systematic distortions

The **least squares method** should be considered whenever **random year-to-year fluctuations** in loss experience are significant.

II. Least Squares Method

Before we dive in, we need to define a few things:

- $x =$ losses to date
- $y =$ losses at a future evaluation

Assuming we have a historical loss triangle, we should have a number of historical (x, y) pairs. The **goal is to predict** y based on x. Let $L(x)$ be the estimate of y, given that we have already observed x.

Link Ratio Method

The link ratio method (i.e., chain-ladder method) **estimates y** as follows:

$$\boxed{L(x) = cx}$$

where c is the selected link ratio. In this paper, $c = \frac{\bar{y}}{\bar{x}}$, which is equal to the volume-weighted average LDF.

Budgeted Loss Method

In some cases, it might make sense to ignore the losses to date. For example:

- When fluctuation in loss experience is extreme
- When past data is not available

In these cases, we could **estimate** y using the budgeted loss method as follows:

$$\boxed{L(x) = k}$$

where k is a constant.

The constant k could be chosen by averaging y over several years, or by multiplying earned premium by an expected loss ratio. For the purposes of this paper, we will assume that k is based on averaging y over several years.

Least Squares Method

The least squares method **estimates** y by fitting a line to the points (x, y) that minimizes the sum of the squares of the residuals. Mathematically:

$$\boxed{L(x) = a + bx}$$

where $b = \frac{\overline{xy} - \bar{x}\bar{y}}{\overline{x^2} - \bar{x}^2}$ and $a = \bar{y} - b\bar{x}$.

Many common methods are **special cases** of the least squares method:

- When $a = 0$, then $L(x) = bx$ (link ratio method)
- When $b = 0$, then $L(x) = a$ (budgeted loss method)
- When $b = 1$, then $L(x) = a + x$ (BF method)

The ability to flex to other methods is a **major advantage** of the least squares method. Later on, we will show that the least squares method is a credibility-weighted average between the link ratio method and the budgeted loss method, highlighting its ability to give more or less weight to the observed value of x as appropriate.

Example: Basic Least Squares Method

An actuary would like to predict AY 2023 losses at 27 months. Given the following information for a small state as of December 31, 2023:

Incurred Losses ($) as of XX Months:

AY	15 mo.	27 mo.	15-27 Link Ratios
2017	19,039	23,279	1.223
2018	33,040	41,560	1.258
2019	14,637	18,937	1.294
2020	2,785	5,185	1.862
2021	51,606	54,206	1.050
2022	5,726	15,726	2.746
2023	40,490		

Due to the volatility in the data and link ratios, we may not want to give full credibility to the high observed loss for 2023 by applying a large link ratio to it. Since the data fluctuations do not appear to be systematic, we should consider using the least squares method.

First, let's calculate the least squares parameters:

Using the LINEST Function

- The known x-values are (19039, 33040, …, 5726)
- The known y-values are (23279, 41560, …, 15726)
- We can use the **LINEST** function in Excel to find the parameters as follows: LINEST(known y values, known x values) = LINEST((23279, 41560, …, 15726), (19039, 33040, …, 5726)) = (0.96781, 6023.70787)
- The first value output by Excel is b and the second value output is a
- Hence, $\boldsymbol{b = 0.96781}$ and $\boldsymbol{a = 6023.70787}$

Using the Least Squares Formulas

On the *exam*, we recommend using the LINEST function to calculate the parameters. However, to demonstrate the least squares formulas, here's how to calculate the parameters by hand:

- $\overline{xy} = \frac{19{,}039(23{,}279)+33{,}040(41{,}560)+\cdots+5{,}726(15{,}726)}{6} = 832{,}562{,}381$
- $\bar{x} = \frac{19{,}039+33{,}040+\cdots+5{,}726}{6} = 21{,}139$
- $\bar{y} = \frac{23{,}279+41{,}560+\cdots+15{,}726}{6} = 26{,}482$
- $\overline{x^2} = \frac{19{,}039^2+33{,}040^2+\cdots+5{,}726^2}{6} = 728{,}681{,}571$
- $\boldsymbol{b} = \frac{\overline{xy}-\bar{x}\bar{y}}{\overline{x^2}-\bar{x}^2} = \frac{832{,}562{,}381-21{,}139(26{,}482)}{728{,}681{,}571-21{,}139^2} = \mathbf{0.96781}$
- $\boldsymbol{a} = \bar{y} - b\bar{x} = 26{,}482 - 0.96781(21{,}139) = \mathbf{6023.70787}$
- These are the same parameters we obtained using the LINEST function

Second, let's estimate the AY 2023 incurred losses at 27 months using the least squares method:

Using the FORECAST Function

- We can use the **FORECAST** function in Excel to estimate y as follows: FORECAST(x, known y values, known x values) = FORECAST(40490, (23279, 41560, …, 15726), (19039, 33040, …, 5726)) = **45210.4966**
- Thus, the estimated AY 2023 incurred losses at 27 months using the least squares method are $45,210.50
- If the only goal is estimate y using the least squares method, then the FORECAST function is useful because it doesn't require the least squares parameters as inputs. However, *we recommend* calculating the parameters to make sure they are sensible (more on this later)

Using the Least Squares Formula

- Recall that $L(x) = a + bx$
- Thus, $L(40{,}490) = 6023.70787 + 0.96781(40{,}490) = \mathbf{45210.4966}$

- This is the same prediction we obtained using the FORECAST function

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Parameter Estimation Errors

In the example above, there was nothing unusual about the least squares parameters. However, both **significant changes in the nature of the loss experience** and **sampling error** can lead to values of a and b that do not reflect reality:

- When $\boldsymbol{a < 0}$, our estimate of y will be negative for small values of x. In this case, use the **link ratio method**
- When $\boldsymbol{b < 0}$, our estimate of y decreases as x increases. In this case, use the **budgeted loss method**

The parameter estimation errors above are why we recommend always calculating the least squares parameters on the *exam*. In the event that $a < 0$ or $b < 0$, you should estimate y using either the link ratio method or budgeted loss method, respectively.

Comparing the Methods Graphically

The following plot compares the link ratio, budgeted loss, and least squares methods for the basic least squares example above: