Rising Fellow

2026

Exam 9

Study Guide

ADVANCED RISK MANAGEMENT

Comprehensive study guide
with original and past CAS problems

Exam 9 Study Guide

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Table of Contents

Introduction

How to Use This Guide

This guide is intended to **supplement** the Content Outline readings. Although we believe it provides a thorough review of the exam material, the readings provide additional context that is invaluable. Please do NOT skip the Content Outline readings.

Past CAS Exam Problems

Past CAS exam problems & solutions are included for each paper. Note that these questions are solely owned by the CAS. They are included in the online course for student convenience. All past CAS problems are Excel-based and can be downloaded from the online course.

Rounding

Numerous examples are provided throughout the guide. We did *not round* any of the intermediate steps in the examples. All calculations were performed with full precision to ensure accuracy, and only the final answers were rounded when necessary. To exactly reproduce the examples, we recommend working them out in Microsoft Excel.

Feedback

We always working to improve the Exam 9 Study Guide and the rest of the Rising Fellow study material. Please send us an email at **exam9@risingfellow.com** if you have feedback about any of the following:

- Sections that are confusing or could be improved
- Errors (ex. formatting, spelling, calculations, grammar, etc.)

Note that errata will be posted on the Rising Fellow website on an as-needed basis.

Blank Pages

Since many students want a printed copy of the study guide, blank pages have been inserted throughout the guide to ensure that all outlines start on odd pages.

I. Proportional Treaties

A **proportional treaty** is an agreement between a reinsurer and a ceding company (i.e., primary insurer, reinsured, cedant). Proportional treaties come in different flavors:

- Quota Share
  - The reinsurer receives a flat percentage of the premium and assumes that same flat percentage of the losses
  - The reinsurer pays the cedant a ceding commission which reflects differences in incurred underwriting expenses
- Surplus Share
  - The reinsurer assumes a part of the risk in proportion to the amount that the insured value exceeds the retained line, up to a given limit. The given limit is expressed as a multiple of the retained line
  - Surplus share treaties are common for property business

Example: Surplus Share Treaty

Suppose an insurer enters into a surplus share treaty with a reinsurer. The contract has the following terms:

- Retained Line: $100,000
- 1st Surplus: 4 lines (4 × $100,000 = $400,000)

Given the following for each risk subject to the treaty:

Risk	Insured Value ($)	Gross Loss & ALAE ($)
1	50,000	30,000
2	100,000	20,000
3	250,000	240,000
4	500,000	25,000
5	1,000,000	500,000
6	10,000,000	200,000

First, let's calculate the 1st surplus % (i.e., the ceded loss %):

Risk	Insured Value ($)	Retained ($)	Reinsured ($)	1st Surplus %
1	50,000	50,000	0	0%
2	100,000	100,000	0	0%
3	250,000	100,000	150,000	60%
4	500,000	100,000	400,000	80%
5	1,000,000	100,000	400,000	40%
6	10,000,000	100,000	400,000	4%

In the table above:

- The maximum reinsured portion is the "number of lines × retained line." In this case, the maximum reinsured portion is 4 × $100,000 = $400,000
- For risks 1 – 2, the insured value is less than or equal to the retained line of $100,000. Thus, the reinsured portion is $0
- For risks 3 – 4, the insured value minus the retained line is less than or equal to the maximum reinsured portion. Thus, the reinsured portion is the insured value minus the retained line. For example, the reinsured portion of risk 3 is $250,000 – $100,000 = $150,000
- For risks 5 – 6, the insured value minus the retained line is greater than the maximum reinsured portion. Thus, the reinsured portion is the maximum reinsured portion of $400,000. Similar to the above, Clark shows the "retained" portion of the insured value as

$100,000 for these two risks. In reality, the insurer is on the hook for the amount of insured value not reinsured ($600,000 for risk 5 and $9,600,000 for risk 6)

- For each risk, the 1st surplus % is equal to $\frac{\text{Reinsured Portion}}{\text{Insured Value}}$. For example, the 1st surplus % for risk 3 is $\frac{150{,}000}{250{,}000} = 60\%$

Second, let's calculate the total ceded loss and total retained loss under the treaty:

Risk	1st Surplus %	Gross Loss & ALAE ($)	Ceded Loss & ALAE ($)
1	0%	30,000	0
2	0%	20,000	0
3	60%	240,000	144,000
4	80%	25,000	20,000
5	40%	500,000	200,000
6	4%	200,000	8,000
Total		**1,015,000**	**372,000**

For each risk, the ceded loss & ALAE is equal to the gross loss & ALAE multiplied by the 1st surplus %. The total ceded loss is $372,000. The total retained loss is $1,015,000 – $372,000 = $643,000.

This example highlights an important point about a surplus share treaty. It is ***not*** excess insurance. The retained line (and associated line limit) informs the reinsured portion and ceded percentage of a given risk. Once the ceded percentage is calculated, the reinsurer is responsible for that percentage of any loss from that risk.

Pricing Proportional Treaties

When **pricing proportional treaties**, the following steps should be taken:

1) **Compile the historical experience on the treaty**:
    - If available, obtain the historical premium and incurred losses on the treaty for five or more years

- If not available, obtain the historical premiums and incurred losses on a gross basis and "adjust the experience" as if the surplus share treaty were in effect
- If the treaty is on a "losses occurring" basis (i.e., aggregate by loss occurrence year), earned premium and accident year losses should be used
- If the treaty is on a "risks attaching" basis (i.e., aggregate by policy written year), written premium and the losses covered by those policies should be used

2) **Exclude catastrophe and shock losses**:
   - Cat losses are due to a single event that impacts a large number of risks (e.g., hurricane)
   - Shock losses are any other losses that distort the overall results. Shock losses typically only impact a single risk
3) **Adjust experience to ultimate level and project to future period**:
   - *First*, we develop losses to their ultimate level
   - *Second*, we adjust historical premiums to the future level. This involves "on-leveling" to bring the historical premiums to the current level. If the premium base is inflation-sensitive (e.g., insured value), then an exposure inflation factor would also need to be included when adjusting historical premiums
   - *Third*, we adjust historical losses to the future level using a loss trend analysis
4) **Select the expected non-catastrophe loss ratio for the treaty**:
   - Assuming the data used in step 3 is reliable, then the expected non-cat loss ratio is equal to the average of the historical loss ratios adjusted to the future level
   - We might want to compare the selected ratio to the cedant's gross CY experience and to industry averages
5) **Load the expected non-catastrophe loss ratio for catastrophes**:
   - Historically, reinsurers had priced catastrophe loads by "spreading" large losses over expected payback periods. This approach may still be used for casualty events but not for property events
   - In today's world, property cat loads are based on catastrophe models that incorporate the risk profile of the cedant

6) **Estimate the combined ratio given the ceding commission and other expenses:**
    - The "other expenses" include the reinsurer's general expenses and overhead, as well as brokerage fees

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Example: Pricing a Proportional Treaty

Given the following historical AY experience and relevant factors for an insurer as of 9/30/22:

AY	Earned Premium ($)	Incurred Losses & ALAE ($)	On-Level Factor	LDF
2017	1,640,767	925,021	1.096	1.000
2018	1,709,371	2,597,041	1.086	1.000
2019	1,854,529	1,141,468	1.034	1.000
2020	1,998,751	1,028,236	0.992	1.000
2021	2,015,522	999,208	1.023	1.075
2022	1,550,393	625,830	1.028	1.600
Total	**10,769,333**	**7,316,804**		

A reinsurer's actuary would like to quote a property quota share treaty effective 1/1/2023 to the insurer. The following information also applies:

- The treaty is to be written on a losses occurring basis
- Selected loss trend = 4%
- Inflation = 3%
- The historical AY 2018 incurred losses & ALAE include $1,582,758 due to a hurricane. There are no other catastrophe losses in the historical period
- A ceding commission of 30% has been suggested by the cedant
- Other expenses for the reinsurer are as follows (as a % of premium):
    - Brokerage fees = 5%
    - Administrative expenses = 1%
    - Unallocated expenses = 1%

First, let's exclude catastrophe losses from the historical losses:

AY	Unadjusted Incurred Losses & ALAE ($)	Non-Cat Incurred Losses & ALAE ($)
2017	925,021	925,021
2018	2,597,041	1,014,283 = 2,597,041 – 1,582,758
2019	1,141,468	1,141,468
2020	1,028,236	1,028,236
2021	999,208	999,208
2022	625,830	625,830
Total	**7,316,804**	**5,734,046**

Second, let's develop and project non-cat losses:

AY	Non-Cat Incurred Losses & ALAE ($)	Trend Factor	LDF	Non-Cat Incurred Losses & ALAE ($)
2017	925,021	1.265	1.000	1,170,152
2018	1,014,283	1.217	1.000	1,234,382 = 1,014,283(1.217)(1.000)
2019	1,141,468	1.170	1.000	1,335,518
2020	1,028,236	1.125	1.000	1,156,766
2021	999,208	1.082	1.075	1,162,229
2022	625,830	1.040	1.600	1,041,381
Total	**5,734,046**			**7,100,427**

In the table above, the trend factor is calculated as $(1.040)^{2023-AY}$. For example, the trend factor for AY 2018 is $(1.040)^{2023-2018} = 1.040^5 = 1.217$.

Third, let's annualize, on-level, and project the earned premium:

AY	EP ($)	Annualization Factor	Inflation Factor	On-Level Factor	On-Leveled, Projected EP ($)
2017	1,640,767	1.000	1.194	1.096	2,147,147
2018	1,709,371	1.000	1.159	1.086	2,151,541
2019	1,854,529	1.000	1.126	1.034	2,159,198
2020	1,998,751	1.000	1.093	0.992	2,167,158
2021	2,015,522	1.000	1.061	1.023	2,187,654

2022	1,550,393	1.333	1.030	1.028	2,188,824
Total	**10,769,333**				**13,001,522**

In the table above:

- The inflation factor is calculated as $(1.03)^{2023-AY}$
- Recall that AY 2022 is a partial year since the historical data is evaluated as of 9/30/22. On the loss side, the LDF for AY 2022 both develops and annualizes the losses. To ensure we are comparing apples to apples, we need to annualize the partial year AY 2022 premium. Since AY 2022 is 75% exposed, the annualization factor for the premium is $\frac{1}{0.75} = 1.333$
- The on-leveled, projected earned premium is found by multiplying the raw earned premium by the three factors

Fourth, let's calculate the expected non-cat loss ratio and load it for cats:

- Using the adjusted premium and losses, the expected non-cat loss ratio is $\frac{7{,}100{,}427}{13{,}001{,}522} = 54.6\%$
- If the cat losses from Hurricane Andrew had been included, the loss ratio would have been approximately 15% higher. As a result, we select a cat loading of 10% (as a % of premium). Note that it would be better to select a cat loading based on a cat model
- The projected loss ratio (including the cat loading) is $54.6\% + 10\% = 64.6\%$. We will use 65%

Fifth, let's calculate the projected combined ratio for the treaty:

- Projected combined ratio = projected loss ratio + ceding commission + other reinsurer expenses. Thus, the projected combined ratio is $65\% + 30\% + 5\% + 1\% + 1\% = 102\%$
- In these assumptions, the treaty appears to be unprofitable (combined ratio $> 100\%$). The reinsurer's actuary might recommend a lower ceding commission as a result

Next, we will discuss the following special features of proportional treaties:

- Sliding scale commission
- Profit commission
- Loss corridor

Sliding Scale Commission

Unlike a flat ceding commission, a sliding scale commission is a percent of premium paid by the reinsurer to the cedant that ***slides*** with the actual loss experience, subject to set minimum and maximum amounts.

Example: Sliding Scale Commission

Given the following commission terms:

- Minimum commission: 25% at a 65% loss ratio
- Sliding 1:1 to: 35% at a 55% loss ratio
- Sliding 0.5:1 to a maximum: 45% at a 35% loss ratio

Based on these terms, the final commissions for various actual loss ratios are as follows:

Actual Loss Ratio	Sliding Scale Commission
30% or below	45.0%
35%	45.0%
40%	42.5%
45%	40.0%
50%	37.5%
55%	35.0%
60%	30.0%
65%	25.0%

Let's discuss the table above:

- An **actual loss ratio at 35% or below** results in a maximum commission of 45%. Hence, the first two rows have a sliding scale commission of 45%
- An **actual loss ratio of 40%** sits between a loss ratio of 35% and a loss ratio of 55%. Thus, the sliding scale commission must sit between 35% and 45%. To calculate the sliding scale commission, we do $commission_{higher} + slide \times (L/R_{higher} - L/R_{actual}) = 35\% + 0.5(55\% - 40\%) = 42.5\%$. The notation "$commission_{higher}$" refers to the commission at the higher loss ratio of the range. The higher loss ratio in this range is 55% and the commission at a 55% loss ratio is 35%. Actual loss ratios of 45% and 50% are calculated in the same way
- An **actual loss ratio of 55%** clearly maps to a sliding scale commission of 35%
- An **actual loss ratio of 60%** sits between a loss ratio of 55% and 65%. Thus, the sliding scale commission must sit between 25% and 35%. To calculate the sliding scale commission, we do $commission_{higher} + slide \times (L/R_{higher} - L/R_{actual}) = 25\% + (65\% - 60\%) = 30\%$
- An **actual loss ratio of 65%** clearly maps to a sliding scale commission of 25%

In the prior example, we determined the adjusted commission *after* the actual loss ratio was known. When pricing a proportional treaty, we must *treat the loss ratio as a random variable*. Thus, for pricing purposes, we are interested in the expected loss ratio.

Now, it may be tempting to simply map the expected loss ratio to the adjusted commission as we did in the prior example. But this fails to consider the fact that the expected loss ratio is an average of all possible outcomes. Hence, we need to consider each possible loss ratio separately, calculate the adjusted commission for each of those possible loss ratios, and then take the average to get an expected commission. Clark provides *two approaches* for doing this:

1) Estimate the expected commission based on the historical loss ratios (including cat and shock losses), adjusted to the future level. This is *not the preferred approach* since it may be distorted by historical cats or years with low premium volume
2) Use an aggregate loss distribution model to create a probability distribution for the loss ratio. This is the *preferred approach*

Example: Expected Sliding Scale Commission

Given the following results from an aggregate loss distribution model:

Range of Loss Ratios	Average in Range	Probability Loss Ratio is in Range
0% – 35%	31.5%	0.025
35% – 55%	46.9%	0.311
55% – 65%	59.9%	0.222
65% or above	82.2%	0.442
Total	**65.0%**	**1.000**

First, we calculate the sliding scale commission for each range using the same sliding scale from the prior example:

- **0% – 35%**: The average loss ratio of 31.5% clearly maps to the maximum commission of 45% since 31.5% < 35%
- **35% – 55%**: The average loss ratio of 46.9% sits between a loss ratio of 35% and a loss ratio of 55%. Thus, the sliding scale commission must sit between 35% and 45%. To calculate the sliding scale commission, we do $commission_{higher} + slide \times (L/R_{higher} - L/R_{actual}) = 35\% + 0.5(55\% - 46.9\%) = 39.1\%$
- **55% – 65%**: The average loss ratio of 59.9% sits between a loss ratio of 55% and a loss ratio of 65%. Thus, the sliding scale commission must sit between 25% and 35%. To calculate the sliding scale commission, we do $commission_{higher} + slide \times (L/R_{higher} - L/R_{actual}) = 25\% + (65\% - 59.9\%) = 30.1\%$

- **65% or above**: The average loss ratio of 82.2% clearly maps to the minimum commission of 25% since $82.2\% > 65\%$

Second, we calculate the expected sliding scale commission using the range probabilities:

- The expected commission is $45.0\%(0.025) + 39.1\%(0.311) + 30.1\%(0.222) + 25\%(0.442) = 31.0\%$

Once we have the expected commission, we can calculate the expected ***technical ratio***, which is the sum of the expected loss ratio and the expected commission. **For this example**, the expected technical ratio is $65.0\% + 31.0\% = 96.0\%$. Note that the expected loss of 65% is found by calculating the sum-product of the average loss ratios for each range and the corresponding probabilities (i.e., the expected value). Of course, in this example, it was provided to you in the data table.

Sliding Scale Commissions & Carryforward Provisions

We can complicate the sliding scale commission calculation by allowing ***carryforward provisions***. Suppose that past loss ratios have been above the loss ratio corresponding to the minimum commission. A carryforward provision allows the "excess loss amount" to be included with the current year's loss in the estimate of the current year's commission. The ***intent of the carryforward provision*** is to smooth results over time.

There are ***two approaches*** for pricing the impact of carryforward provisions:

1) Estimate the impact on the current year only
    - With this approach, we shift the slide by the amount of the carryforward
    - The issue with this approach is that it ignores the potential for carryforward beyond the current year
2) Estimate the impact on a block of years (e.g., the next five years)
    - With this approach, the variance of the aggregate distribution is reduced since we assume that individual bad years will be smoothed by individual good years

- The reduced variance is captured by putting higher probabilities in the ranges closer to the expected loss ratio
- **One issue** with this approach is that the method for reducing the variance is not obvious
- **Another issue** with this approach is that it ignores the possibility that the contract may not renew the following year

Let's look at an example of the first approach for incorporating a carryforward provision.

Example: Sliding Scale Commissions & Carryforward Provisions

Given the following tables from an aggregate loss distribution model:

Range of Loss Ratios	Average in Range	Probability Loss Ratio is in Range
0% – 35%	31.5%	0.025
35% – 55%	46.9%	0.311
55% – 65%	59.9%	0.222
65% or above	82.2%	0.442
Total	**65.0%**	**1.000**

Range of Loss Ratios	Average in Range	Probability Loss Ratio is in Range
0% – 30%	27.4%	0.006
30% – 50%	43.0%	0.221
50% – 60%	55.1%	0.222
60% or above	78.3%	0.551
Total	**65.0%**	**1.000**

Assume the following:

- The treaty includes a carryforward provision
- The cedant's loss ratio in the prior year for the book of business underlying the treaty was 70%

- The sliding scale commission terms are as follows:
  - Minimum commission: 25% at a 65% loss ratio
  - Sliding 1:1 to: 35% at a 55% loss ratio
  - Sliding 0.5:1 to a maximum: 45% at a 35% loss ratio

First, we calculate the carryforward provision:

- The loss ratio associated with the minimum commission is 65%. The carryforward provision is the excess loss amount from the prior year. In this case, the carryforward provision is $70\% - 65\% = 5\%$

Second, we calculate the shifted sliding scale based on the carryforward provision:

- Minimum commission: 25% at a 60% loss ratio, where $60\% = 65\% - 5\%$
- Sliding 1:1 to: 35% at a 50% loss ratio, where $50\% = 55\% - 5\%$
- Sliding 0.5:1 to a maximum: 45% at a 30% loss ratio, where $30\% = 35\% - 5\%$

Third, we calculate the sliding scale commission for each range using the sliding scale terms:

- We must use the second table produced by the aggregate loss distribution model since the ranges match the shifted sliding scale ranges
- **0% – 30%**: The average loss ratio of 27.4% clearly maps to the maximum commission of 45% since $27.4\% < 30\%$
- **30% – 50%**: The average loss ratio of 43.0% sits between a loss ratio of 30% and a loss ratio of 50%. Thus, the sliding scale commission must sit between 35% and 45%. To calculate the sliding scale commission, we do $commission_{higher} + slide \times (L/R_{higher} - L/R_{actual}) = 35\% + 0.5(50\% - 43.0\%) = 38.5\%$
- **50% – 60%**: The average loss ratio of 55.1% sits between a loss ratio of 50% and a loss ratio of 60%. Thus, the sliding scale commission must sit between 25% and 35%. To calculate the sliding scale commission, we do $commission_{higher} + slide \times (L/R_{higher} - L/R_{actual}) = 25\% + (60\% - 55.1\%) = 29.9\%$

- **60% or above**: The average loss ratio of 78.3% clearly maps to the minimum commission of 25% since $78.3\% > 60\%$

Fourth, we calculate the expected sliding scale commission using the range probabilities:

- The expected commission is $45.0\%(0.006) + 38.5\%(0.221) + 29.9\%(0.222) + 25\%(0.551) = 29.2\%$

Fifth, we calculate the expected technical ratio:

- Expected technical ratio = expected loss ratio + expected commission = $65.0\% + 29.2\% = 94.2\%$

Profit Commission

The profit commission is much simpler than the sliding scale commission. Here are the steps for calculating a profit commission **after** the actual loss ratio is known:

- Subtract the actual loss ratio, ceding commission, and a margin for expenses from 1.00 to determine the reinsurer's profit as a percentage of premium
- Return a specified percentage of the reinsurer's profit as a profit commission

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Example: Profit Commission

Given the following:

- Actual loss ratio = 55%
- Ceding commission = 25%
- Margin = 10%
- The reinsurer returns 50% of its profit back to the cedant

The reinsurer's profit is $100\% - 55\% - 25\% - 10\% = 10\%$. Thus, the profit commission is $50\%(10\%) = 5\%$. If the actual ceded premium was $1,000,000, then the dollar amount of the profit commission back to the cedant would be $\$1{,}000{,}000(5\%) = \$50{,}000$.

In this example, we have already observed the actual loss ratio. If we wanted to include the profit commission while pricing the treaty, we would need to calculate an expected profit commission similar to the sliding scale commission examples. In this case, for each range from an aggregate loss distribution model, we would calculate the profit commission based on the average loss ratio in the range. Then, we would calculate the expected profit commission using the range probabilities. An example of this is provided in the Cookbook.

Loss Corridor

Under a loss corridor, the cedant re-assumes a portion of the reinsurer's liability for a specified loss ratio layer.

Example: Loss Corridor

Given the following table from an aggregate loss distribution model:

Range of Loss Ratios	Average in Range	Probability Loss Ratio is in Range
0% – 80%	64.1%	0.650
80% – 90%	84.7%	0.156
90% or above	103.9%	0.194
Total	**75.0%**	**1.000**

The treaty includes a loss corridor with the following terms:

- The cedant re-assumes 75% of the 80% to 90% loss ratio layer

First, let's calculate the reinsurer's loss ratio net of the loss corridor for each loss ratio range:

- **0% − 80%**: Since the average loss ratio of 64.1% is below 80%, the loss corridor does not apply. Thus, the reinsurer's loss ratio net of the loss corridor is still 64.1%
- **80% − 90%**: Since the average loss ratio of 84.7% is above 80%, the loss corridor applies. Under the loss corridor terms, 75% of $(84.7\% - 80\%)$ is re-assumed by the cedant. Thus, the reinsurer's loss ratio net of the loss corridor is $84.7\% - 0.75(84.7\% - 80\%) = 81.2\%$
- **90% or above**: Since the average loss ratio of 103.9% is above 80%, the loss corridor applies. Under the loss corridor terms, 75% of $(90\% - 80\%)$ is re-assumed by the cedant. Thus, the reinsurer's loss ratio net of the loss corridor is $103.9\% - 0.75(90\% - 80\%) = 96.4\%$

Second, let's calculate the reinsurer's expected loss ratio net of the loss corridor using the range probabilities:

- The reinsurer's loss ratio net of the loss corridor is $64.1\%(0.650) + 81.2\%(0.156) + 96.4\%(0.194) = 73.0\%$
- As we can see, the loss ratio reduces the reinsurer's expected loss ratio by two points (from 75% to 73%)
- This example also highlights why we can't simply look at an expected loss ratio. The expected loss ratio before applying the loss corridor is 75%. Even though this is below the corridor layer of $80\% - 90\%$, the reinsurer still benefits because we must consider each possible outcome (not just the expected outcome)

II. Property Per-Risk Excess Treaties

Property per-risk excess treaties apply on a per-risk basis and provide a limit of coverage in excess of the cedant's retention. Some important reminders about property per-risk excess treaties are as follows:

- The treaty premium is set as a percent of a subject premium base
- For losses occurring policies, the subject premium is called the "gross net earned premium income" (GNEPI)
- For risks attaching policies, the subject premium is called the "gross net written premium income" (GNWPI)
- For both policy types, the "net" refers to the fact that the subject premium is net of any other reinsurance inuring to the benefit of the per-risk treaty (i.e., reinsurance that is applied before the per-risk treaty is applied). The "gross" refers to the fact the subject premium is gross of the per-risk treaty being priced

Per-risk excess treaties are priced using either **experience rating** or **exposure rating**.

Experience Rating

The steps for experience rating are as follows:

1) Assemble the subject premium and historical losses for several years
2) Adjust the subject premium to the future level (i.e., on-leveling and inflation)
3) Apply loss trend factors to the historical large losses and determine the amount included in the excess layer being analyzed. If ALAE applies pro-rata with losses, it should be added individually for each loss
4) Apply excess development factors to the summed excess losses for each historical period
5) Divide the trended and developed layered losses by the adjusted subject premium to calculate loss costs for each historical period and in total

If the projected loss costs for each year are increasing or decreasing over time, then the model assumptions are not met and should be re-examined (e.g., the trend or LDFs may be too high or too low, there may have been shifts in the types of business written by the cedant, etc.).

Example: Experience Rating – Property Per-Risk Excess Treaty

A reinsurer's actuary is pricing a property per-risk excess treaty with the following terms as of 12/31/2022:

- Effective Date: 1/1/23
- Treaty Limit: $400,000
- Attachment Point: $100,000
- Loss Trend: 4%
- Exposure Trend: 2%
- The treaty is on a losses occurring basis
- No other reinsurance policies apply to this book of business

The historical subject premium and corresponding on-level factors are as follows:

AY	Gross Net Premium Income ($)	On-Level Factor
2018	1,910,507	1.02
2019	1,936,665	1.03
2020	1,963,617	1.04
2021	2,081,886	1.01

The historical large losses are as follows:

Accident Date	Untrended Total Loss ($)
2/23/18	324,298
4/30/18	100,549
9/22/18	75,475
1/1/19	171,885
5/18/19	94,218
8/19/19	170,297
8/15/20	87,133
7/12/21	771,249

The following excess LDFs apply to the $400,000 xs $100,000 layer:

	12-Ult.	24-Ult.	36-Ult.	48-Ult.	60-Ult.
CDF	2.25	1.30	1.15	1.05	1.01

First, we adjust the historical subject premium (i.e., GNEPI) to the future level:

AY	GNEPI ($)	On-Level Factor	Trend Factor	Adjusted Subject Premium ($)
2018	1,910,507	1.02	1.104	2,151,384
2019	1,936,665	1.03	1.082	2,158,336
2020	1,963,617	1.04	1.061	2,166,734
2021	2,081,886	1.01	1.040	2,186,813

In the table above:

- The trend factors are calculated as $(1 + Exposure\ Trend)^{2023-AY}$. For example, the trend factor for AY 2021 is $(1.02)^{2023-2021} = (1.02)^2 = 1.040$
- The adjusted subject premium is found by multiplying the GNEPI by the on-level and trend factors

Second, we trend the losses and calculate the excess layer losses:

Accident Date	Untrended Total Loss ($)	Years of Trend	Trend Factor	Trended Loss ($)	Treaty Layer Loss ($)
2/23/18	324,298	5.36	1.234	400,184	300,184
4/30/18	100,549	5.17	1.225	123,173	23,173
9/22/18	75,475	4.78	1.206	91,024	0
1/1/19	171,885	4.50	1.193	205,059	105,059
5/18/19	94,218	4.12	1.175	110,706	10,706
8/19/19	170,297	3.87	1.164	198,226	98,226
8/15/20	87,133	2.88	1.119	97,589	0
7/12/21	771,249	1.97	1.080	832,949	400,000

In the table above:

- For losses, we *trend from the accident date to the average prospective accident date*
  - For example, the first loss must be trended from 2/23/18 to 7/1/23
  - The second loss must be trended from 4/30/18 to 7/1/23
  - To calculate the trend period in years, we can use the "YEARFRAC(Start Date, End Date)" function in Excel
  - Thus, the trend period for the 2/23/18 accident is "YEARFRAC(2/23/18, 7/1/23)" = 5.36
  - Then, the trend factor for this loss is $(1 + Loss\ Trend)^{\text{Trend Period}} = (1.04)^{5.36} = 1.234$
- The trended total loss is equal to the untrended total loss multiplied by the trend factor
- The loss in the treaty layer is equal to $min(max(Trended\ Total\ Loss - Attachment\ Point, 0), Treaty\ Limit)$
  - Thus, for the 2/23/18 accident, the loss in the treaty layer is $min(max(400{,}184 - 100{,}000, 0), 400{,}000) = 300{,}184$

Third, we aggregate the treaty layer losses by AY and develop them using the excess LDFs:

AY	Trended Treaty Layer Losses ($)	Excess LDF	Trended, Developed Treaty Layer Losses ($)
2018	323,356	1.01	326,590
2019	213,991	1.05	224,690
2020	0	1.15	0
2021	400,000	1.30	520,000

In the table above, AY 2021 is 24 months old at 12/31/22. Thus, we multiply the aggregated, trended treaty layer losses by the 24-Ultimate excess LDF of 1.30. The other years are calculated in a similar manner.

Mildenhall & Major Ch. 12 – Classical Price Allocation Theory

Outline

In this chapter, we examine classical allocation methods to assist in determining prices for individual units.

I. The Allocation of Portfolio Constant CoC (CCoC) Pricing

The allocation of portfolio CCoC pricing involves four sets of variables: 1) premium P, 2) expected loss l, 3) CoC target return, and 4) assets a. These variables are related by five equations:

1) $P = l + \delta(a - l)$
2) $P_i = l_i + \delta_i(a_i - l_i)$
3) $P = \sum_i P_i$
4) $l = \sum_i l_i$
5) $a = \sum_i a_i$

As suggested by the name, we assume a constant CoC across all individual units. In practice, we "fix" the target return and then allocate assets, which in turn determines the premium. Note that the formula for P above is equivalent to the Ch. 8 formula for $\bar{P} = \frac{E(X)+\iota a}{1+\iota}$, where $\delta = \frac{\iota}{1+\iota}$ and ι is the CoC.

There are *two approaches* to risk-adjust the target return and fix the amount of capital:

1) Risk-adjusted return on capital (**RAROC**) methods that say return varies with risk
2) Return on risk-adjusted capital (**RORAC**) methods that say return is constant when capital reflects risks

The RORAC method, which combines a constant CoC with a capital allocation that normalizes for risk, is the industry practice. We call this the **allocated CCoC pricing method**.

In Chapter 14, we will challenge the concept of a constant CoC. Since insurers are funded by different types of capital with varying cost characteristics, each unit will consume a unit mix of capital. So, even all units have a constant CoC within a layer, the weighted cost of capital across layers will differ since each unit has a different mix of capital.

To ***determine the constant CoC to be used***, we rely on an estimate of the insurer's weighted average cost of capital (between debt, reinsurance, and equity). As mentioned in an earlier chapter, we can quantify debt and reinsurance costs. This leaves the equity cost of capital as the most important unknown input to the weighted average cost of capital.

To ***determine the risk-adjusted capital*** (also known as economic capital or risk capital), there is no widely-accepted practice. Thus, we will look at various classical techniques that have been used. In addition, allocated capital is artificial because the entire capital base of the insurer is available for each individual unit. However, we still need it because it influences the decisions insurers make by unit.

II. Allocation of Non-Additive Functionals

In this section, our goal is to define a function that applies a risk measure function to loss amounts and uses those to allocate some sort of total risk measure to each unit. We call the allocated amount a_i. The a_i does not necessarily have to be allocated assets. We can think of it more generally. It could also be allocated premium. If the attribution is additive (meaning the unit amounts add up to the total amount $\rho(X)$), we call it an ***allocation of*** ρ.

Three desirable properties of an allocation are as follows:

1) It should work at any level of granularity
2) It should be decomposable, which means that the allocation to a sum of random variables equals the sum of their allocations
3) It should be computed using a single, consistent formula

It is possible for the risk measure used for allocation to differ from the risk measure used to calculate the total ρ. When the *same risk measure is used* to determine the total ρ and allocate it, we call it an **endogenous** allocation. When the *same risk measure is not used*, we call it an **exogenous** allocation.

Now, let's look at specific allocation methods.

Expected Value

The expected value allocation works as follows:

$$\boxed{a_i = a(X)\frac{E(X_i)}{E(X)}}$$

In this case, the total amount $a(X)$ is allocated to each unit i in proportion to expected loss. This is an **exogenous** allocation since the allocation is based on expected value.

Proportional Allocation

The proportional allocation (also known as stand-alone pro rata or scaled allocation) is an endogenous allocation that works as follows:

$$\boxed{a_i = a(X)\frac{a(X_i)}{\sum_i a(X_i)}}$$

In this allocation, each unit is evaluated on a stand-alone basis (i.e., the $a(X_i)$) and the total is allocated based on these stand-alone measures. Since it uses stand-alone measures, it is not influenced by the dependence between X_i. It is **endogenous** because the same risk measure used to calculate $a(X)$ is used to calculate the individual $a(X_i)$.

Haircut Allocation

The haircut allocation is an exogenous version of proportional allocation that works as follows:

$$\boxed{a_i = a(X)\frac{\rho(X_i)}{\sum_i \rho(X_i)}}$$

This is **exogenous** because the stand-alone measures are based on a general ρ. Like proportional allocation, it is not influenced by the dependence between X_i.

Equal Risk Allocation

The equal risk allocation or quantile allocation can be written **endogenously** or **exogenously**:

- The ***endogenous*** allocation solves $\sum_i a(X_i, p^*) = a(X)$ for p^* and sets $a_i = a(X_i, p^*)$
- The ***exogenous*** version solves $\sum_i \rho(X_i, p^*) = a(X)$ for p^* and sets $a_i = \rho(X_i, p^*)$

The idea is to capitalize each unit to the same probability of default $1 - p^*$ (hence, "equal risk"). Under the other allocation methods we have discussed, the probability of default differs by unit. This allocation is often used in the Lloyd's/London market.

Example: Classical Allocation Methods – Simple Discrete Case Study

In this example, we will apply the prior allocation methods to the Simple Discrete case study. As we have seen many times before, the Simple Discrete case study has the following setup on a gross basis:

X_1 (\$M)	X_2 (\$M)	X (\$M) $= X_1 + X_2$	$P(X_1)$	$P(X_2)$	$P(X) = P(X_1)P(X_2)$
0	0	0	0.50	0.50	0.2500
0	1	1	0.50	0.25	0.1250
0	90	90	0.50	0.25	0.1250
8	0	8	0.25	0.50	0.1250
8	1	9	0.25	0.25	0.0625
8	90	98	0.25	0.25	0.0625

10	0	10	0.25	0.50	0.1250
10	1	11	0.25	0.25	0.0625
10	90	100	0.25	0.25	0.0625

As mentioned before, the allocation formulas are general. We could use them to allocate assets to each unit and then use those unit assets to calculate unit premiums. Alternatively, we can also allocate total premium to each unit directly. To align with the Ch. 12 examples shown in the textbook, we will use the latter approach. Suppose total premium is $80.

First, let's allocate the total premium of $80 to each unit using the **expected value** allocation:

- $E(X_1) = 4.5$, $E(X_2) = 22.75$, and $E(X) = 27.25$
- $P_1 = P\left(\frac{E(X_1)}{E(X)}\right) = 80\left(\frac{4.5}{27.25}\right) = \$13.21$
- $P_2 = P\left(\frac{E(X_2)}{E(X)}\right) = 80\left(\frac{22.25}{27.25}\right) = \$66.79$

Second, let's allocate the total premium of $80 to each unit using the **haircut** allocation based on individual $\text{TVaR}_{0.75}$ measures:

- $\text{TVaR}_{0.75}(X_1) = 10$
- $\text{TVaR}_{0.75}(X_2) = 90$
- $P_1 = P\left(\frac{\text{TVaR}_{0.75}(X_1)}{\text{TVaR}_{0.75}(X_1)+\text{TVaR}_{0.75}(X_2)}\right) = 80\left(\frac{10}{10+90}\right) = \$8$
- $P_2 = P\left(\frac{\text{TVaR}_{0.75}(X_2)}{\text{TVaR}_{0.75}(X_1)+\text{TVaR}_{0.75}(X_2)}\right) = 80\left(\frac{90}{10+90}\right) = \$72$

Third, let's allocate the total premium of $80 to each unit using the **equal risk** allocation:

- We need to find p^* such that $\text{TVaR}_{p^*}(X_1) + \text{TVaR}_{p^*}(X_2) = 80$
- We recognize that $0.5 \le p^* \le 0.75$ since $\text{TVaR}_{0.50}(X_1) = 9$, $\text{TVaR}_{0.50}(X_2) = 45.5$ and $\text{TVaR}_{0.75}(X_1) = 10$, $\text{TVaR}_{0.75}(X_2) = 90$
- We solve for p^* using our Ch. 4 algorithm for $\text{TVaR}_p(X_i)$

- Hence, $\text{TVaR}_{p^*}(X_1) = \frac{1}{1-p^*}[8(0.75 - p^*) + 10(1.00 - 0.75)]$ and $\text{TVaR}_{p^*}(X_2) = \frac{1}{1-p^*}\big(1(0.75 - p^*) + 90(1.00 - 0.75)\big)$. If we sum these together, set the sum equal to 80, and solve for p^*, we find that $p^* = 0.6796$
    - As we reminder on the TVaR calculation, we are simply calculating the conditional expected value for each unit as demonstrated in the Ch. 4 TVaR algorithm
    - Note that the textbook has an error in the calculations shown at the top of page 315. Although the final allocations shown are correct, the "0.5×10" and "0.5×90" should be "0.25×10" and "0.25×90"
- Then, $\text{TVaR}_{0.6796}(X_1) = \frac{1}{1-0.6796}[8(0.75 - 0.6796) + 0.25(10)] = 9.56$ and $\text{TVaR}_{0.6796}(X_2) = \frac{1}{1-0.6796}[1(0.75 - 0.6796) + 0.25(90)] = 70.44$. Thus, the allocation is $\$80 = \$9.56 + \$70.44$

Marginal Business Unit Allocation

The marginal business unit or Merton-Perold method attributes to each unit the reduction in capital from dropping it from the portfolio. In other words, we compute the risk capital for the total portfolio with and without the business unit. The difference in risk capital is the "allocated capital" for that unit. This method is *not additive*. In general, the sum of the individual unit allocations is less than the total capital.

Marginal Business Euler Gradient Allocation

Unlike the Merton-Perold method, we do not calculate the change in capital from removing an entire business unit with this method. Instead, we calculate the marginal change in capital given a marginal change in the amount of unit i written. This is equivalent to taking the derivative of capital with respect to unit i. Then, we apply this derivative to the unit i total to obtain the allocation. This method is endogenous.

A prior study showed that the Euler allocation is the *only one suitable for performance measurement*. The reason is as follows:

- Suppose we use the Euler allocation to determine the risk capital for each unit i
- Then, suppose we divide the profit for each unit by its risk capital to determine RORAC
- Since we used the Euler allocation, growing (or shrinking) lines with a higher (or lower) RORAC always improves the average return. This is not the case with the other allocations. **As a quick example**, suppose that RORAC for LOB A is 10% and the RORAC for LOB B is 15%. If the RORACs use allocated risk capital based on the Euler allocation, then the insurer can be confident that growing in LOB B and shrinking in LOB A will improve the overall average return

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Example: Marginal Business Euler Gradient Allocation

Suppose an insurer uses a factor-based capital formula with three inputs: 1) net written premium P, 2) net reserves R, and 3) invested assets a. The capital formula is $\rho(P, R, a) = \sqrt{(0.4P)^2 + (0.25R)^2 + (0.10a)^2}$. Further suppose that $P = \$1{,}000$, $R = \$3{,}000$, and $a = \$3{,}500$.

In this example, we are not interested in allocating capital to lines of business. Instead, we want to allocate capital to individual "risks." Let's do this using Euler allocation.

First, let's take the partial derivative of ρ with respect to each input and plug in 1,000 for P, 3,000 for R, and 3,500 for a to obtain the final figures:

- $\frac{\partial \rho}{\partial P} = \frac{1}{2}\left(\frac{2(0.4^2 P)}{\sqrt{(0.4P)^2+(0.25R)^2+(0.10a)^2}}\right) = 0.174$
- $\frac{\partial \rho}{\partial R} = \frac{1}{2}\left(\frac{2(0.25^2 R)}{\sqrt{(0.4P)^2+(0.25R)^2+(0.10a)^2}}\right) = 0.204$
- $\frac{\partial \rho}{\partial a} = \frac{1}{2}\left(\frac{2(0.10^2 a)}{\sqrt{(0.4P)^2+(0.25R)^2+(0.10a)^2}}\right) = 0.038$

Second, let's calculate the allocated capital for each risk by multiplying the derivatives by the total amounts for each unit:

- $a_P = 0.174(1{,}000) = \$174.10$ (rounding)
- $a_R = 0.204(3{,}000) = \$611.90$ (rounding)
- $a_a = 0.038(3{,}500) = \$133.30$ (rounding)

If we sum these allocated capital amounts up, we get \$919.20, which also equals $\rho(P, R, a)$. Thus, the allocation method is ***additive***.

In addition, this allocation produces significantly different risk charges than the stand-alone factors from the model. **For example**, the factor for premium risk is clearly 0.40. If we apply this to the premium total, we obtain a charge of \$400. But this is much higher than the Euler allocation of \$174.1 In other words, the factors produce a premium risk charge that is too high.

Game Theory and the Shapley Allocation

The Shapley allocation is based on game theory. In a "game," business units can join together or stand alone. The central premise is that we want an allocation such that no unit is allocated more than it would incur going it alone. This is because there is no incentive in joining others if there is no benefit from doing so. An allocation where no unit is allocated more than its stand-alone costs satisfies the **no-undercut property**.

Now, let's formally define a "game:"

- There are n units
- There is a cost function c on subsets of $\{1, 2, \ldots, n\}$. Suppose S is a subset of $\{1, 2, \ldots, n\}$. Then, $c(S)$ is the cost of operating the units in S together
- The game is called **atomic** when each unit is completely in or completely out
- The game is called **fractional** when units can form fractional coalitions (like a quota-share)

- The set of allocations that satisfies the no-undercut property is called the **core** of the game

The **Shapley value or allocation** is the ideal allocation when a game is atomic. The Shapley allocation to each unit i is notated as c_i and is defined as follows:

$$c_i = \sum_{S \subset \{1,\ldots,n\}, i \notin S} \frac{|S|!\,(n-|S|-1)!}{n!}\big(c(S \cup \{i\}) - c(S)\big)$$

where $|S|$ denotes the number of elements in S.

The Shapley allocation has the following *desirable qualities*:

- It is additive
- It is symmetric. If two units i and j increase the cost of every S that contains neither i nor j by the same amount (i.e., $c(S \cup \{i\}) = c(S \cup \{j\})$), then $c_i = c_j$
- It is linear in game theory
- It is homogenous if c is
- If c is sub-additive, then the Shapley value satisfies the no-undercut property
- It allocates no capital to a constant risk

The Shapley allocation suffers from *two drawbacks*:

- To allocate to n units, we must compute 2^n marginal impacts, which is impractical
- If a unit is sub-divided further into two new units, then allocations assigned to the other units change

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Example: Shapley Allocation – Two-Player Game

Suppose we want to derive the Shapley allocations for a two-player game. In this case, our total portfolio set is $\{1,2\}$.

First, find the Shapley allocation for player 1:

- There are two subsets of $\{1,2\}$ that do not include 1: $S = \{\emptyset\}$ (i.e., the empty set) and $S = \{2\}$
- Next, we must apply our formula for $c_1 = \sum_{S \subset \{1,2\}, 1 \notin S} \frac{|S|!(n-|S|-1)!}{n!}\big(c(S \cup \{1\}) - c(S)\big)$. We summarize the various pieces of this formula in the following table:

S	$|S|$	**Increment** = $c(S \cup \{1\}) - c(S)$
$\emptyset$	0	$c(1) - c(\emptyset) = c(1) - 0 = c(1)$
$\{2\}$	1	$c(1,2) - c(2)$

- Since $n = 2$, $c_1 = \frac{0!(2-0-1)!}{2!}\big(c(1)\big) + \frac{1!(2-1-1)!}{2!}\big(c(1,2) - c(2)\big) = 0.5c(1) + 0.5\big(c(1,2) - c(2)\big)$

Second, find the Shapley allocation for player 2:

- Using the same approach as c_1, $c_2 = \frac{0!(2-0-1)!}{2!}\big(c(2)\big) + \frac{1!(2-1-1)!}{2!}\big(c(1,2) - c(1)\big) = 0.5c(2) + 0.5\big(c(1,2) - c(1)\big)$

If we sum the allocations together, we get $0.5c(1) + 0.5\big(c(1,2) - c(2)\big) + 0.5c(2) + 0.5\big(c(1,2) - c(1)\big) = c(1,2)$. Thus, the Shapley allocations for each player sum up to the total portfolio cost of $c(1,2)$.

A *nice feature* of a two-player game is that the Shapley allocation for each unit is the average of the stand-alone capital for the unit and the Merton-Perold marginal capital of removing the unit from the total. **For example**, for unit 1, the stand-alone capital is $c(1)$ and the Merton-Perold marginal benefit of removing unit 1 from the total is $c(1,2) - c(2)$. Thus, $c_1 = 0.5\big[c(1) + \big(c(1,2) - c(2)\big)\big]$.

Example: Shapley Allocation – Three-Player Game

Suppose we want to derive the Shapley allocations for a three-player game. In this case, our total portfolio set is $\{1,2,3\}$.

Let's find the Shapley allocation for player 1:

- There are four subsets of $\{1,2,3\}$ that do not include 1: $S = \{\emptyset\}$, $S = \{2\}$, $S = \{3\}$, and $S = \{2,3\}$
- Next, we must apply our formula for $c_1 = \sum_{S\subset\{1,2,3\},1\notin S} \frac{|S|!(n-|S|-1)!}{n!}\big(c(S \cup \{1\}) - c(S)\big)$. We summarize the various pieces of this formula in the following table:

S	$\lvert S \rvert$	**Increment** = $c(S \cup \{1\}) - c(S)$
$\emptyset$	0	$c(1) - c(\emptyset) = c(1) - 0 = c(1)$
$\{2\}$	1	$c(1,2) - c(2)$
$\{3\}$	1	$c(1,3) - c(3)$
$\{2,3\}$	2	$c(1,2,3) - c(2,3)$

- Since $n = 3$, $c_1 = \frac{0!(3-0-1)!}{3!}\big(c(1)\big) + \frac{1!(3-1-1)!}{3!}\big(c(1,2) - c(2)\big) + \frac{1!(3-1-1)!}{3!}\big(c(1,3) - c(3)\big) + \frac{2!(3-2-1)!}{3!}\big(c(1,2,3) - c(2,3)\big) = \frac{1}{3}c(1) + \frac{1}{6}\big(c(1,2) - c(2)\big) + \frac{1}{6}\big(c(1,3) - c(3)\big) + \frac{1}{3}\big(c(1,2,3) - c(2,3)\big)$

The Shapley allocations for player 2 and player 3 work in the exact same way. If we sum up the Shapley allocations, they will equal the total portfolio cost of $c(1,2,3)$.

Example: Shapley Allocation – Simple Discrete Case Study

In this example, we apply the Shapley allocation to the Simple Discrete case study. Once again, we have the following on a gross basis:

X_1 (\$M)	X_2 (\$M)	X (\$M) $= X_1 + X_2$	$P(X_1)$	$P(X_2)$	$P(X) = P(X_1)P(X_2)$
0	0	0	0.50	0.50	0.2500
0	1	1	0.50	0.25	0.1250
0	90	90	0.50	0.25	0.1250
8	0	8	0.25	0.50	0.1250
8	1	9	0.25	0.25	0.0625
8	90	98	0.25	0.25	0.0625
10	0	10	0.25	0.50	0.1250
10	1	11	0.25	0.25	0.0625
10	90	100	0.25	0.25	0.0625

Assume the cost function c is $\text{TVaR}_{0.75}$.

First, let's calculate the Shapley allocation for unit X_1:

- $c(1) = \text{TVaR}_{0.75}(X_1) = 10$
- $c(2) = \text{TVaR}_{0.75}(X_2) = 90$
- $c(1,2) = \text{TVaR}_{0.75}(X) = \frac{0.125(90)+0.0625(98)+0.0625(100)}{1-0.75} = 94.5$
- Since this is a two-player game, $c_1 = 0.5\left[c(1) + \left(c(1,2) - c(2)\right)\right] = 0.5(10 + 94.5 - 90) = 7.25$
- In summary, for unit X_1, the stand-alone measure is 10, the marginal (i.e., Merton-Perold) allocation is 94.5 – 90 = 4.5, and the Shapley allocation is 7.25

Second, let's calculate the Shapley allocation for unit X_2:

- Since this is a two-player game, $c_2 = 0.5\left[c(2) + \left(c(1,2) - c(1)\right)\right] = 0.5(90 + 94.5 - 10) = 87.25$
- In summary, for unit X_2, the stand-alone measure is 90, the marginal (i.e., Merton-Perold) allocation is 94.5 – 10 = 84.5, and the Shapley allocation is 87.25

Example: Shapley Allocation with the Standard Deviation PCP

Ins Co. writes three units A, B, and C with the following characteristics:

- Losses from each unit are normally distributed with a mean of 1,000, and with CVs of 10%, 20%, and 30%, respectively
- Units A and B are 50% correlated, A and C are 40% correlated, and B and C are 30% correlated

Ins Co. prices using the standard deviation PCP, where premium equals expected loss plus 50% of standard deviation.

First, let's derive the premiums for the seven subsets of A, B, and C. The following table summarizes the premiums:

Portfolio	Expected Loss	Standard Deviation	Premium
A	1,000	100	1,050.00
B	1,000	200	1,100.00
C	1,000	300	1,150.00
AB	2,000	264.58	2,132.29
AC	2,000	352.14	2,176.07
BC	2,000	407.43	2,203.72
ABC	3,000	469.04	3,234.52

Here are some of the calculations for the table above:

- $SD(A) = 1{,}000(0.10) = 100$
- $SD(AB) = \sqrt{100^2 + 200^2 + 2(100)(200)(0.50)} = 264.58$
- $SD(ABC) = \sqrt{100^2 + 200^2 + 300^2 + 2(100)(200)(0.50) + 2(100)(300)(0.40) + 2(200)(300)(0.30)} = 469.04$
- $P_A = 1{,}000 + 0.50(100) = 1{,}050.00$

- $P_{AB} = 2{,}000 + 0.50(264.58) = 2{,}132.29$
- $P_{ABC} = 3{,}000 + 0.50(469.04) = 3{,}234.52$

Second, let's calculate the Shapley allocation for unit A:

- There are four subsets of $\{A, B, C\}$ that do not include A: $S = \{\emptyset\}$, $S = \{B\}$, $S = \{C\}$, and $S = \{B, C\}$
- Next, we must apply our formula for $c_A = \sum_{S \subset \{A,B,C\}, A \notin S} \frac{|S|!(n-|S|-1)!}{n!}\big(c(S \cup \{A\}) - c(S)\big)$, where the cost function c is the premium PCP being used by Ins Co. We summarize the various pieces of this formula in the following table:

S	$\lvert S \rvert$	**Increment =** $P_{S \cup \{A\}} - P_S$
$\emptyset$	0	$P_A = 1{,}050.00$
$\{B\}$	1	$P_{AB} - P_B = 1{,}032.29$
$\{C\}$	1	$P_{AC} - P_C = 1{,}026.07$
$\{B, C\}$	2	$P_{ABC} - P_{BC} = 1{,}030.81$

- Since $n = 3$, $c_A = \frac{0!(3-0-1)!}{3!}(P_A) + \frac{1!(3-1-1)!}{3!}(P_{AB} - P_B) + \frac{1!(3-1-1)!}{3!}(P_{AC} - P_C) + \frac{2!(3-2-1)!}{3!}(P_{ABC} - P_{BC}) = 1{,}036.66$

The Shapley allocations for unit B and unit C work the same way.

The last thing covered in this section is the Aumann-Shapley allocation. Given its technical nature, we will not cover it in the guide.

Co-Measure Allocations

If a risk measure ρ can be written in the form $\rho(X) = E[h(X)L(X)]$ where h is an additive function, then the **co-measure** $E[h(X_i)L(X)]$ can be used as an allocation to the i^{th} unit. Notice that only the $h(\,)$ function takes in X_i as an input. The $L(\,)$ function is still based on the total X.